

SINGULAR SOLUTIONS OF THE $(3 + 1)$ -D PROTTER PROBLEM FOR THE WAVE EQUATION

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Abstract: We study some boundary value problems for the nonhomogeneous wave equation with three space and one time variables. The problems could be viewed as $\mathbb{R}^4$ analogs of Darboux problems in $\mathbb{R}^2$. In contrast to the planar Darboux problem the four-dimensional version is ill-posed, since its homogeneous adjoint problem has infinitely many classical solutions. Thus, in the framework of the classical solvability the problem is not Fredholm. Alternatively, it is known that for smooth right-hand side functions, there is a uniquely determined generalized solution that may have strong power-type singularity at one boundary point. This singularity is isolated at the vertex of the characteristic light cone and does not propagate along the cone. In this article we give a general existence result and find a priori estimates for singular solutions. A lengthy reference list is appended.

Keywords: wave equation, boundary value problems, generalized solution, singular solutions, propagation of singularities, special functions.

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Аннотация. Изучаются некоторые краевые задачи для неоднородного волнового уравнения с тремя пространственными и одной временной переменными. Эти задачи можно рассматривать как четырехмерный аналог плоской задачи Дарбу. В отличие от плоской задачи четырехмерная задача оказывается некорректной, поскольку однородная сопряженная к ней задача имеет бесконечно много классических решений. Таким образом, в рамках классической теории изучаемая задача не является фредгольмовой. С другой стороны, известно, что для гладкой правой части существует однозначно определенное обобщенное решение, которое может иметь сильную особенность в граничной точке. Эта особенность будет изолирована в вершине характеристического светового конуса и не будет распространяться вдоль конуса. В работе доказывается общий результат о существовании, существование сингулярных решений и для них устанавливаются априорные оценки. Прилагается обширная библиография.

Ключевые слова: волновое уравнение, краевая задача, обобщенное решение, сингулярное решение, распространение сингулярности, специальные функции.

1. Introduction

We study some boundary value problems for the wave equation in $\mathbb{R}^4$ that were proposed by Murrey Protter in the 1950s. Consider the wave equation with three space and one time variables

$$u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} - u_{tt} = f(x, t) \quad (1)$$

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for $(x, t) = (x_1, x_2, x_3, t) \in \mathbb{R}^4$ in the domain

$$\Omega = \{(x, t) : 0 < t < 1/2, \ t < \sqrt{x_1^2 + x_2^2 + x_3^2} < 1 - t\}.$$

This domain is bounded by the two characteristic cones

$$\Sigma_1 = \{(x, t) : 0 < t < 1/2, \ \sqrt{x_1^2 + x_2^2 + x_3^2} = 1 - t\},$$

$$\Sigma_2 = \{(x, t) : 0 < t < 1/2, \ \sqrt{x_1^2 + x_2^2 + x_3^2} = t\}$$

and the ball

$$\Sigma_0 = \{t = 0, \ \sqrt{x_1^2 + x_2^2 + x_3^2} < 1\},$$

centered at the origin O ; i.e., $x = 0$ and $t = 0$. The right-hand side function f of (1) satisfies some smoothness conditions in Ω that will be fixed later. We will study the following BVPs:

Problem P1. *Find a solution of (1) in Ω satisfying the boundary conditions*

$$P1 : \quad u|_{\Sigma_0} = 0, \quad u|_{\Sigma_1} = 0.$$

Problem P1*. *Find a solution of (1) in Ω satisfying the adjoint boundary conditions*

$$P1^* : \quad u|_{\Sigma_0} = 0, \quad u|_{\Sigma_2} = 0.$$

In this paper we give a general existence result and discuss the behavior of the generalized solution of Problem P1.

First, we present a brief historical overview here and provide an extensive list of references. Protter arrived at these problems while examining BVPs for mixed type equations which describe transonic flows in fluid dynamics. In particular, the classical two-dimensional Guderley–Morawetz problem for the Gellerstedt equation of hyperbolic-elliptic type which models flows around airfoils. The Guderley–Morawetz problem is well studied in the 1950s—see the surveys for the 2-D mixed type BVPs and the transonic models by Morawetz [1, 2]. For example, the existence of weak solutions and the uniqueness of strong solutions were proved by Morawetz [3], while Lax and Phillips [4] showed that the weak solutions are strong. In 1954 Protter [5, 6] formulated for 3D mixed-type equations (with two space and one time variables) some multidimensional analogs of the planar Guderley–Morawetz problem. The assumption was that the methods used to attack the 2D case could be applied for the multi-dimensional problems. However, the multi-dimensional case turns out to be rather different and the situation there is still unclear. Although, the uniqueness of the so-called quasiregular solutions is proved by Aziz and Schneider in [7], there are no general existence results for the Protter mixed-type problems. Even the question of well-posedness is not resolved completely.

As regards the results for existence or nonexistence of nontrivial solutions of related quasilinear problems of mixed hyperbolic-elliptic type in the multidimensional case, see [8, 9]. About results on BVPs for the multidimensional mixed-type Lavrent'ev–Bitsadze equation, see [10, 11].

In relation to the mixed-type problems, Protter also formulated and studied in [6] some BVPs in the hyperbolic part of the domain both for degenerated hyperbolic equations and for the wave equation—the 3D variants of Problems $P1$ and $P1^*$. Later Paul Garabedian [12] gave the statement of such problems in $\mathbb{R}^4$ and proved the uniqueness of the classical solutions of Problem $P1$. Problems $P1$ and $P1^*$ in Ω could be considered as four-dimensional analogs of the planar Darboux problems (or the Cauchy–Goursat problems) for the string equations in a characteristic triangle. Initially, the expectation was that such multidimensional BVPs are classically solvable for very smooth right-hand side functions. Contrary to this traditional belief, soon it became clear that unlike the planar Darboux problem, the Protter problems are ill-posed. In fact, the homogeneous adjoint problem $P1^*$ has smooth classical solutions and the linear space they span is infinite-dimensional. Thus, in the frame of the classical solvability the Protter problem $P1$ is not Fredholm, since it has infinite-dimensional cokernel. Alternatively, the notion of generalized solution that may have singularity on Σ_2 was introduced in [28]. In fact, it is known that the generalized solution has singularity isolated at only one point—the origin O . The point O lies both on the characteristic part of the boundary Σ_2 and on the noncharacteristic part Σ_0 , and this case is different from the standard propagation of singularities (see Hörmander [13, Chapter 24.5]). A short survey and comparison of various recent results for Protter problems are in [14, 15]. In [16] the semi-Fredholm solvability of Problem $P1$ is discussed. According to the classical and singular solutions let us mention here some results of Serik Aldashev in [10, 11, 17, 18] and a series of papers of Khe Kan Cher [19–22] and also the joint papers with his co-authors [23, 24].

Results for the wave equation but with lower order terms could be found in [25, 26]. Regarding results for degenerated hyperbolic equations we refer to [18, 27, 28], and for equations of Keldysh type—to [29]. Some other multidimensional analogs of the classical Darboux problem are considered in [30–33].

In the present article we give sufficient conditions on the right-hand function f for the existence of a generalized solution of Problem $P1$ and discuss its exact behavior. On the one hand, we need a priori estimates away from the origin to ensure the existence of the solution (like in Theorems 1 and 2). On the other hand, our goal is to study singularity near O . We find upper estimates for the growth the generalized solution at O in Theorem 2 and Corollary 1, and a lower estimate is given in Theorem 3.

2. Classical and Generalized Solutions

In order to construct the solutions of the homogeneous adjoint problem $P1^*$ we need in $\mathbb{R}^3$ the orthonormal system of the *spherical functions* Y_n^m ($n \in \mathbb{N} \cup \{0\}$, and $m = 1, \dots, 2n + 1$). They are defined usually on the unit sphere $S^2 := \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\}$ in spherical polar coordinates (see [34]). Expressed in Cartesian coordinates here, we can define them by

$$Y_n^{2k}(x_1, x_2, x_3) = C_{n,k} \frac{d^k}{dx_3^k} P_n(x_3) \operatorname{Im}\{(x_1 + ix_2)^k\}, \quad \text{for } k = 1, \dots, n; \quad (2)$$

$$Y_n^{2k+1}(x_1, x_2, x_3) = C_{n,k} \frac{d^k}{dx_3^k} P_n(x_3) \operatorname{Re}\{(x_1 + ix_2)^k\}, \quad \text{for } k = 0, \dots, n,$$

where $C_{n,k}$ are constants and P_n are the *Legendre polynomials*. The Legendre polynomials are defined by the Rodrigues formula as

$$P_n(s) := \frac{1}{2^n n!} \frac{d^n}{ds^n} (s^2 - 1)^n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} a_{n,2k} s^{n-2k},$$

with the coefficients

$$a_{n,2k} = (-1)^k \frac{(2n-2k)!}{2^n k! (n-k)! (n-2k)!}.$$

The constants $C_{n,m}$ are such that Y_n^m form a complete orthonormal system in $L_2(S^2)$ (see [34]). For convenience in the discussions that follow, we extend the spherical functions beyond S^2 radially, keeping the same notation Y_n^m for the extended functions, i.e., $Y_n^m(x) := Y_n^m(x/|x|)$ for $x \in \mathbb{R}^3 \setminus O$.

Let us define, for $k \in \mathbb{N} \cup \{0\}$, the functions

$$h_k(\xi, \eta) = \int_{\eta}^{\xi} s^k P_n \left(\frac{\xi\eta + s^2}{s(\xi + \eta)} \right) ds.$$

Then Lemma 1.1 and Lemma 2.3 from [35] give the following solutions of the homogeneous adjoint problem.

Lemma 1 [35]. *The functions*

$$v_{k,m}^n(x, t) = |x|^{-1} h_{n-2k-2} \left(\frac{|x|+t}{2}, \frac{|x|-t}{2} \right) Y_n^m(x).$$

are classical solutions from $C^\infty(\Omega) \cap C(\overline{\Omega})$ of the homogeneous problem $P1^*$ for $n \in \mathbb{N}$, $m = 1, \dots, 2n+1$ and $k = 0, 1, \dots, \lfloor (n-1)/2 \rfloor - 2$.

Actually, the functions $v_{k,m}^n$ here are practically the same as the solutions from [35, Lemma 1.1]. On the other hand, [16, Theorem 1] suggests that there are no other linearly independent nontrivial classical solutions of the homogenous adjoint Problem $P1^*$. Solutions for the homogenous adjoint problem were first found by Tong Kwang-Chang [39]. Some different representations of the solutions of the homogeneous problem $P1^*$ and the functions $v_{k,m}^n$ are given by Khe Kan Cher [22].

Naturally, a necessary condition for the existence of a classical solution for Problem P1 is the orthogonality of the right-hand side function f to all $v_{k,m}^n(x, t)$. To avoid infinitely many necessary conditions in the framework of the classical solvability, we introduce *generalized solutions* for Problem P1, eventually with singularity at the origin O .

DEFINITION 1 [36]. A function $u = u(x, t)$ is called a *generalized solution* of Problem P1 in Ω , if the following are satisfied:

- 1) $u \in C^1(\overline{\Omega} \setminus O)$, $u|_{\Sigma_0 \setminus O} = 0$, $u|_{\Sigma_1} = 0$, and
- 2) we have

$$\int_{\Omega} (u_t w_t - u_{x_1} w_{x_1} - u_{x_2} w_{x_2} - u_{x_3} w_{x_3} - f w) dx dt = 0 \quad (3)$$

for all $w \in C^1(\overline{\Omega})$ such that $w = 0$ on Σ_0 and a neighborhood of Σ_2 .

Here we find some appropriate conditions for f under which there exists a generalized solution of Problem P1.

3. Existence of a Generalized Solution

The spherical functions form a complete orthonormal system in $L_2(S^2)$, and generally, each smooth function $f(x, t)$ can be expanded as a harmonic series

$$f(x, t) = \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} f_n^m(|x|, t) Y_n^m(x) \quad (4)$$

with the Fourier coefficients

$$f_n^m(r, t) := \int_{S(r)} f(x, t) Y_n^m(x) d\sigma_r, \quad (5)$$

where $S(r)$ is the three-dimensional sphere in $x = (x_1, x_2, x_3)$ variables; i.e., $S(r) := \{x \in \mathbb{R}^3 : |x| = r\}$. In the previous paper [36], the Protter problem was studied in the special case when the right hand side function is a finite Fourier sum, while in [16] for the general case $f \in C^1(\overline{\Omega})$ the necessary and sufficient conditions for the existence of bounded solutions were found. In fact, the behavior of the generalized solution depends strongly on the inner product (with respect to the $L_2(\Omega)$ inner product) of the right-hand side function $f(x, t)$ with the functions $v_{k,m}^n(x, t)$ from Lemma 1. Thus, let us denote by $\beta_{k,m}^n$ the parameters

$$\beta_{k,m}^n := \int_{\Omega} v_{k,m}^n(x, t) f(x, t) dx dt, \quad (6)$$

where $n = 0, \dots, l$, $k = 0, \dots, [\frac{n-1}{2}]$ and $m = 1, \dots, 2n+1$.

Theorem 1 [16]. *Let $f(x, t)$ belong to $C^{10}(\overline{\Omega})$. Then the necessary and sufficient conditions for existence of a bounded generalized solution $u(x, t)$ of Protter's Problem P1 are*

$$\int_{\Omega} v_{k,m}^n(x, t) f(x, t) dx dt = 0, \quad (7)$$

for all $n \in \mathbb{N}$, $k = 0, \dots, [\frac{n-1}{2}]$, $m = 1, \dots, 2n+1$.

Moreover, this generalized solution $u(x, t) \in C^1(\overline{\Omega} \setminus O)$ and satisfies the a priori estimates

$$|u(x, t)| \leq C \|f\|_{C^{10}(\overline{\Omega})}, \quad (8)$$

$$\sum_{i=1}^3 |u_{x_i}(x, t)| + |u_t(x, t)| \leq C(|x|^2 + t^2)^{-1} \|f\|_{C^{10}(\overline{\Omega})} \quad (9)$$

where C is a constant independent of $f(x, t)$.

We turn now to investigation of the singular solution of Problem P1.

In order to formulate a general existence result, let us introduce, for $p \in \mathbb{R}$ and $k \in \mathbb{N}$, the series

$$\|f; n^p; C^k\| := \|f_0^0(|x|, t)\|_{C^0(\Omega)} + \sum_{n=1}^{\infty} n^p \left\| \sum_{m=1}^{2n+1} f_n^m(|x|, t) Y_n^m(x) \right\|_{C^k(\Omega)}$$

and the power series

$$\Phi(s) := \sum_{n=1}^{\infty} \left[\sum_{m=1}^{2n+1} \sum_{k=0}^{[n/2]} |\beta_{k,m}^n| \right] s^n.$$

Obviously, $\|f; n^{p_1}; C^{k_1}\| \geq \|f; n^{p_2}; C^{k_2}\|$ for $p_1 \geq p_2$ and $k_1 \geq k_2$.

Now we can formulate the main result in the present paper.

Theorem 2. Let $f(x, t)$ belong to $C^1(\overline{\Omega})$. Suppose that the series $\|f; n^6; C^0\|$ and $\|f; n^4; C^1\|$ are convergent and the power series $\Phi(s)$ has an infinite radius of convergence. Then there exists a unique generalized solution $u(x, t) \in C^1(\overline{\Omega} \setminus O)$ of Protter's Problem P1 and it satisfies in $\overline{\Omega} \setminus O$ the a priori estimates

$$\begin{aligned} |u(x, t)| &\leq C \left[\Phi \left(\frac{C_1}{|x| + t} \right) + |x|^{-1} \|f; n^4; C^0\| \right], \\ |u(x, t)| &\leq C \left[\Phi \left(\frac{C_1}{|x| + t} \right) + \|f; n^6; C^0\| + \|f; n^4; C^1\| \right], \\ \sum_{i=1}^3 |u_{x_i}(x, t)| + |u_t(x, t)| &\leq C |x|^{-2} \left[\Phi \left(\frac{C_2}{|x| + t} \right) + \|f; n^6; C^0\| \right], \end{aligned}$$

where C , C_1 , and C_2 are constants independent of $f(x, t)$.

In these estimates, the singularity of the generalized solution at O is controlled by the function $\Phi(s)$, while $\|f; n^p; C^k\|$ bounds the “regular part” of $u(x, t)$.

Let us compare the situation here for the (3+1)-D case (three space and one time dimensions) with the results of [37] for (2+1)-D Protter Problems. According to Theorem 5.3 of [37], the sufficient condition for existence of generalized solution is the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{n} I_0 \left(\frac{2n}{\varepsilon} \right) (\|f_n^1\|_{C^0(\Omega)} + \|f_n^2\|_{C^0(\Omega)}), \quad \text{for all } \varepsilon > 0, \quad (10)$$

where f_n^i are the Fourier coefficients of the right-hand side which could be viewed as the analogs of the functions f_n^m given by (5). The function I_0 is the modified Bessel function of the first kind:

$$I_0(s) := \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{s}{2} \right)^{2k}.$$

Notice that we have the estimate

$$I_0(s) \leq e^s \quad \text{for } s \geq 0.$$

Using the exponential function in (10) instead of I_0 , we then get the following result somewhat weaker than [37, Theorem 5.3]. Suppose that the power series

$$\Phi_1(s) := \sum_{n=1}^{\infty} (\|f_n^1\|_{C^0(\Omega)} + \|f_n^2\|_{C^0(\Omega)}) n^{-1} s^n$$

converges for all s . Then for the singularity of the unique generalized solution $u(x, t)$ for the (2+1)-D Protter Problem, near the origin we have the estimate

$$|u(x, t)| \leq C \Phi_1 \left[\exp \left(\frac{2}{|x|} \right) \right]. \quad (11)$$

Here, in the (3+1)-D case, note that from the definition of $v_{k,m}^n$ in Lemma 1 it follows that

$$|v_{k,m}^n(x, t)| \leq |Y_n^m(x)| \leq C_1 n^{1/2}$$

and so

$$|\beta_{k,m}^n| \leq C_2 n^{1/2} \|f_n^m\|_{C^0(\Omega)}.$$

Then, to compare with [37, Theorem 5.3] and (11), we can formulate the next result that ensues from Theorem 2.

Corollary 1. *Let $f(x, t)$ belong to $C^1(\overline{\Omega})$. Suppose that the series $\|f; n^6; C^0\|$ and $\|f; n^4; C^1\|$ are convergent and the power series*

$$\Phi_2(s) := \sum_{n=1}^{\infty} \left[\sum_{m=1}^{2n+1} \|f_n^m\|_{C^0(\Omega)} \right] s^n$$

has an infinite radius of convergence. Then the unique generalized solution $u(x, t) \in C^1(\overline{\Omega} \setminus O)$ of Protter's Problem P1 satisfies near the origin the estimate

$$|u(x, t)| \leq C \Phi_2 \left(\frac{C_0}{|x| + t} \right) \quad (12)$$

where C and C_0 are constants independent of $f(x, t)$.

4. Construction of Singular Solutions

In the special case when the right-hand side function f is a harmonic polynomial, the exact asymptotic formula for the generalized solution at O is found in [36]. It shows that the solution can have only power type singularity. However, in the general case $f(x, t) \in C^1(\overline{\Omega})$ some stronger singularities are also possible. Actually, in [38] the existence of generalized solutions with at least exponential growth at the origin is announced. The next theorem could be used to construct other singular solutions.

Theorem 3. *Let $f(x, t)$ belong to $C^1(\overline{\Omega})$, while the series $\|f; n^6; C^0\|$ and $\|f; n^4; C^1\|$ are convergent, and the power series $\Phi(s)$ has an infinite radius of convergence. Let the numbers $\alpha_p \geq 0$, $p = 0, 1, 2, \dots$, be such that the series*

$$\phi(s) := \sum_{p=0}^{\infty} \alpha_p s^p$$

converges for all $s \in \mathbb{R}$. Suppose that there is $x^ = (x_1^*, x_2^*, x_3^*) \in \mathbb{R}^3$ such that*

$$\sum_{k=0}^{\infty} \sum_{m=1}^{2p+4k+1} p a_{2k} \beta_{m,k}^{p+2k} Y_{p+2k}^m(x^*) \geq \alpha_p. \quad (13)$$

Then there exist a number $\delta \in (0, 1/2)$ that the unique generalized solution $u(x, t)$ of Problem P1 satisfies the estimate

$$|u(tx_1^*, tx_2^*, tx_3^*, t)| \geq \phi \left(\frac{1}{2t} \right)$$

for $t \in (0, \delta)$.

REMARK. We can find a right-hand side $f(x, t) \in C^1(\overline{\Omega})$ by choosing suitable Fourier coefficients $f_n^m(r, t)$ “small enough” so that the required series $\|f; n^p 4; C^k\|$ and $\Phi(s)$ be convergent. At the same time, selecting the functions f_n^m that satisfy (13) with larger constants α_p will produce solutions with a stronger singularity. In accordance with the result from [38], it is possible to obtain an appropriate function f with constants $\alpha_p = (p!)^{-1}$ for all p , and so the corresponding solution has exponential growth at O .

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