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ON A $3D$ ELASTIC BODY WITH A THIN RIGID INCLUSION

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Abstract: We analyze an equilibrium state of a three-dimensional elastic body with a thin two-dimensional inclusion. At the first step we assume the inclusion to be delaminated from the surrounding elastic body, which provides an interfacial crack. To describe a mutual nonpenetration between the crack faces, we impose inequality type boundary conditions at the crack faces. Additional difficulties appear in view of the Neumann type conditions considered at the external boundary of the elastic body. We analyze a passage to the limit provided that the stiffness parameter of the inclusion tends to infinity. The limit model is analyzed, which describes an equilibrium state of the elastic body with a thin rigid inclusion. Various equivalent problem formulations of the limit problem are presented.

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Keywords: elastic body, crack, Neumann and inequality type boundary conditions, thin rigid inclusion.

Dedicated to Ivan Egorovich Egorov
on the occasion of his 75th birthday

1. Thin elastic inclusion

In the paper [1], a boundary value problem describing an equilibrium state of a three-dimensional elastic body with a thin delaminated elastic inclusion was analyzed. The Neumann boundary conditions at the external boundary of the elastic body was imposed. The goal of this paper is to substantiate the possibility of limiting the stiffness parameter of the thin inclusion within the framework of the considered model and, in the limit, to obtain a thin rigid inclusion incorporated into the elastic body.

A huge number of publications can be found related to the equilibrium states of elastic bodies with thin incorporated elastic, rigid and semirigid inclusions [1–17], see also [18, 19]. Almost all publications describe $2D$ elastic bodies with $1D$ inclusions. In the mentioned works, the discussions concern a solvability of boundary value problems, asymptotic analysis of solutions, optimal control problems, junction problems, shape sensitivity of solutions, inverse problems, etc. As for boundary

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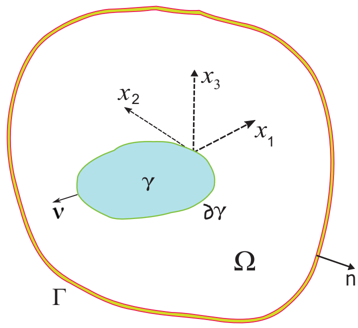

Fig. 1. Elastic body with inclusion γ.

problems for elastic bodies with Neumann conditions imposed on external boundaries, we refer the reader to [20–22] and the literature therein. The present paper focuses on justification of a passage to the limit with respect to the stiffness parameter, providing a transition from delaminated elastic inclusion to a rigid one for $3D$ elastic body with the Neumann condition considered at the external boundary.

First, we present the formulation of the problem analyzed in [1] for a $3D$ elastic body with a thin elastic inclusion.

Denote by $\Omega \subset \mathbb{R}^3$ a bounded domain with smooth boundary Γ; $\overline{\gamma} \subset \Omega$, $\gamma = \gamma_0 \times \{0\}$, γ_0 is a bounded $2D$ domain with a smooth boundary, see Fig. 1. Let $n = (n_1, n_2, n_3)$ be a unit normal vector to Γ; $\nu = (\nu_1, \nu_2)$ is the unit vector to the boundary $\partial\gamma_0$ of γ_0; $\Omega_\gamma = \Omega \setminus \overline{\gamma}$.

Introduce elasticity tensors $A = \{a_{ijkl}\}$, $i, j, k, l = 1, 2, 3$; $B = \{b_{ijkl}\}$, $D = \{d_{ijkl}\}$, $i, j, k, l = 1, 2$, satisfying the usual properties of symmetry and positive definiteness, $a_{ijkl} \in L^\infty(\Omega)$; $b_{ijkl}, d_{ijkl} \in L^\infty(\gamma)$.

All functions with two lower indices are symmetric in those indices. Summation convention over repeated indices is used; functions defined on γ are identified with functions of the variable $(x_1, x_2) \in \gamma_0$.

For a scalar function w defined on γ we put $\nabla\nabla w = \{w_{,ij}\}$, $i, j = 1, 2$. If $M = \{M_{ij}\}$, $i, j = 1, 2$, then $\nabla\nabla M = M_{ij,ij}$. Introduce notations for a bending moment M^ν and a transverse force $T^\nu = T^\nu(M)$,

$$M^\nu = -M_{ij}\nu_j\nu_i; \quad T^\nu = -M_{ij,j}\nu_i - M_{ij,k}\tau_k\tau_j\nu_i, \quad (\tau_1, \tau_2) = (-\nu_2, \nu_1). \tag{1.1}$$

In the model analyzed, the domain Ω_γ represents an elastic body, and γ corresponds to a thin elastic inclusion.

The formulation of the problem, considered in [1], is as follows. We have to find a displacement field $u = (u_1, u_2, u_3)$ of the elastic body and a displacement $v = (v_1, v_2), w$ of the thin inclusion defined on γ such that

$$-\operatorname{div}\sigma = g, \sigma = A\varepsilon(u) \text{ in } \Omega_\gamma, \tag{1.2}$$

$$-\operatorname{div} N = [(\sigma_{13}, \sigma_{23})], \quad N = B\varepsilon(v) \text{ on } \gamma, \tag{1.3}$$

$$-\nabla\nabla M = [\sigma_{33}], \quad M = -D\nabla\nabla w \text{ on } \gamma, \tag{1.4}$$

$$\sigma n = 0 \text{on} \Gamma; \quad N\nu = 0, \quad M^\nu = T^\nu = 0 \text{ on } \partial\gamma, \tag{1.5}$$

$$u_i^- = v_i, \ i = 1,2, \quad u_3^- = w \text{ on } \gamma, \tag{1.6}$$

$$[u_3] \geq 0, \quad \sigma_{33}^+ \leq 0, \quad \sigma_{i3}^+ = 0, \ i = 1,2, \quad [u_3]\sigma_{33}^+ = 0 \text{ on } \gamma, \tag{1.7}$$

$$\int\limits_{\Omega_\gamma} u = 0, \quad \int\limits_{\Omega_\gamma} (u_{i,j} - u_{j,i}) = 0, \quad i,j = 1,2,3. \tag{1.8}$$

Here, the function $g \in L^2(\Omega)^3$ represents a given external force acting on the elastic body; $[h] = h^+ - h^-$ is a jump of a function h on γ; $N = N(v) = \{N_{ij}\}$, $M = M(w) = \{M_{ij}\}$, $i,j = 1,2$, are stress and moment tensors to be defined on γ; $\sigma = \sigma(u) = \{\sigma_{ij}\}$, $i,j = 1,2,3$, is the stress tensor for the elastic body to be defined in Ω_γ; $\varepsilon(u) = \{\varepsilon_{ij}(u)\}$, $\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i})$ is the strain tensor, $i,j = 1,2,3$. The inclusion γ is delaminated from the elastic material, forming a crack between γ and the elastic body. In particular, the delamination of the elastic inclusion means that a displacement field of the elastic body has a jump on γ. The signs $\pm$ correspond to positive and negative faces of the crack with respect to the normal vector $(0,0,1)$. It is assumed that the delamination takes place at γ^+. In such a case, suitable glue conditions are imposed at γ^-, see (1.6). Relations (1.2) are the equilibrium equation of the elastic body and the constitutive law (Hooke's law), respectively. Relations (1.3), (1.4) provide equilibrium and constitutive law equations for the thin inclusion γ being an elastic plate in the frame of Kirchhoff-Love approach. As for boundary conditions (1.7), they guarantee a mutual non-penetration between the crack faces $\gamma^\pm$ with zero friction at γ^+. The contact set is unknown a priori in the frame of the model (1.2)–(1.8); the approach used corresponds to a free boundary one. The right-hand sides in the first equations of (1.3) and (1.4) describe forces acting on γ from the surrounding elastic body. The first boundary condition of (1.5) corresponds to the Neumann type which implies the non-coercivity of the problem (1.2)–(1.7). To obtain a unique solution, we impose conditions (1.8).

The problem (1.2)–(1.7) admits a variational formulation. To this end, consider Sobolev spaces $H^1(\Omega_\gamma), H^i(\gamma)$, $i = 1,2$,

$$U = \{(u,v,w) \in H^1(\Omega_\gamma)^3 \times H^1(\gamma)^2 \times H^2(\gamma) \mid u_i^- = v_i, \ i = 1,2,$$
$$u_3^- = w \text{ on } \gamma; \ u = (u_1,u_2,u_3), \ v = (v_1,v_2)\}$$

and the subspace $U_1 \subset U$,

$$U_1 = \left\{(u,v,w) \in U \mid \int\limits_{\Omega_\gamma} u = 0, \int\limits_{\Omega_\gamma} (u_{i,j} - u_{j,i}) = 0, \ i,j = 1,2,3; \ u = (u_1,u_2,u_3)\right\}.$$

For any set $G \subset \mathbb{R}^3$, introduce the space of infinitesimal rigid displacements

$$R(G) = \{\rho = (\rho_1,\rho_2,\rho_3) \mid \rho(x) = C + Fx, \ x \in G; \ C = (c^1,c^2,c^3),$$
$$F = \{f_{ij}\}, \ f_{ij} = -f_{ji}, \ i,j = 1,2,3; \ c^i, f_{ij} \in \mathbb{R}\}.$$

Note that if $(\rho_1, \rho_2, \rho_3) \in R(\gamma)$, we have

$$(\rho_1, \rho_2)(x) = (c^1, c^2) + f(x_2, -x_1), \quad \rho_3(x) = c^3 + f_1 x_1 + f_2 x_2;$$

$$x = (x_1, x_2) \in \gamma_0; \quad c^i, f, f_i \in \mathbb{R}.$$

Consider next the energy functional $E : U \to \mathbb{R}$,

$$E(u, v, w) = \frac{1}{2} \int\limits_{\Omega_\gamma} \sigma(u)\varepsilon(u) - \int\limits_{\Omega_\gamma} gu + \frac{1}{2} \int\limits_{\gamma} N(v)\varepsilon(v) - \frac{1}{2} \int\limits_{\gamma} M(w)\nabla\nabla w.$$

For simplicity we write $\sigma(u)\varepsilon(u)$ and $N(v)\varepsilon(v)$ instead of $\sigma_{ij}(u)\varepsilon_{ij}(u)$ and $N_{ij}(v)\varepsilon_{ij}(v)$, respectively.

Let Q be the set of admissible displacements

$$Q = \{(u, v, w) \in U \mid [u_3] \geq 0 \text{ on } \gamma; u = (u_1, u_2, u_3)\}.$$

Consider the minimization problem

$$\inf_{(u,v,w) \in Q \cap U_1} E(u, v, w). \tag{1.9}$$

It is proved in [1] that the problem (1.9) has a unique solution (u, v, w) provided that the external forces satisfy the following condition:

$$\int\limits_{\Omega_\gamma} g\rho = 0 \quad \forall \rho \in R(\Omega_\gamma). \tag{1.10}$$

This solution satisfies the variational inequality

$$(u, v, w) \in Q \cap U_1, \tag{1.11}$$

$$\int\limits_{\Omega_\gamma} \sigma(u)\varepsilon(\overline{u}-u) - \int\limits_{\Omega_\gamma} g(\overline{u}-u) + \int\limits_{\gamma} N(v)\varepsilon(\overline{v}-v) - \int\limits_{\gamma} M(w)\nabla\nabla(\overline{w}-w) \geq 0 \; \forall (\overline{u}, \overline{v}, \overline{w}) \in Q. \tag{1.12}$$

Moreover, the problem formulations (1.11), (1.12) and (1.2)–(1.8) are equivalent assuming that the solutions are quite smooth. This means that all relations (1.11), (1.12) can be derived from (1.11), (1.12), and conversely.

We should add that the condition (1.10) is not only sufficient for the solvability of the problem (1.11), (1.12) but it is necessary.

2. Stiffness parameter tends to infinity

In this section, we introduce a positive parameter δ into the model (1.2)–(1.8). This parameter characterizes a stiffness of the inclusion γ. Our aim is to justify a passage to the limit as $\delta \to \infty$. More precisely, supplying the displacement fields with the parameter δ, we consider the boundary value problem for finding $u = \left(u_1^\delta, u_2^\delta, u_3^\delta\right)$, $v^\delta = \left(v_1^\delta, v_2^\delta\right)$, w^δ, such that

$$-\operatorname{div} \sigma = g, \quad \sigma = A\varepsilon(u^\delta) \text{ in } \Omega_\gamma, \tag{2.1}$$

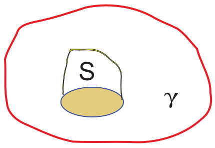

Fig. 2. Neighborhood S.

$$-\delta \operatorname{div} N = [(\sigma_{13}, \sigma_{23})], \quad N = B\varepsilon(v^\delta) \text{ on } \gamma, \tag{2.2}$$

$$-\delta \nabla\nabla M = [\sigma_{33}], \quad M = -D\nabla\nabla w^\delta \text{ on } \gamma, \tag{2.3}$$

$$\sigma n = 0 \text{ on } \Gamma; \quad N\nu = 0, \quad M^\nu = T^\nu = 0 \text{ on } \partial\gamma, \tag{2.4}$$

$$u_i{}^{\delta -} = v_i^\delta, \ i = 1,2, \quad u_3{}^{\delta -} = w^\delta \text{ on } \gamma, \tag{2.5}$$

$$[u_3^\delta] \geq 0, \sigma_{33}^+ \leq 0, \quad \sigma_{i3}^+ = 0, \ i = 1,2, \quad [u_3^\delta]\sigma_{33}^+ = 0 \text{ on } \gamma, \tag{2.6}$$

$$\int\limits_{\Omega_\gamma} u^\delta = 0, \quad \int\limits_{\Omega_\gamma} (u_{i,j}^\delta - u_{j,i}^\delta) = 0, \ i,j = 1,2,3. \tag{2.7}$$

The condition (1.10) is assumed to be fulfilled in this section.

The problem (2.1)–(2.7) admits a variational statement and corresponds to minimization of the energy functional

$$E_\delta(u,v,w) = \frac{1}{2}\int\limits_{\Omega_\gamma} \sigma(u)\varepsilon(u) - \int\limits_{\Omega_\gamma} gu + \frac{\delta}{2}\int\limits_{\gamma} N(v)\varepsilon(v) - \frac{\delta}{2}\int\limits_{\gamma} M(w)\nabla\nabla w$$

over the set $Q \cap U_1$. The solution of this problem satisfies the following variational inequality

$$(u^\delta, v^\delta, w^\delta) \in Q \cap U_1, \tag{2.8}$$

$$\int\limits_{\Omega_\gamma} \sigma(u^\delta)\varepsilon(\overline{u} - u^\delta) - \int\limits_{\Omega_\gamma} g(\overline{u} - u^\delta) + \delta\int\limits_{\gamma} N(v^\delta)\varepsilon(\overline{v} - v^\delta)$$
$$- \delta\int\limits_{\gamma} M(w^\delta)\nabla\nabla(\overline{w} - w^\delta) \geq 0 \quad \forall(\overline{u}, \overline{v}, \overline{w}) \in Q. \tag{2.9}$$

At the first step, we obtain a priori estimates for the solution of the problem (2.8), (2.9) being uniform with respect to δ. These estimates will allow us to pass to the limit in (2.8), (2.9) as $\delta \to \infty$.

From (2.8), (2.9) the following relation is obtained

$$\int\limits_{\Omega_\gamma} \sigma(u^\delta)\varepsilon(u^\delta) - \int\limits_{\Omega_\gamma} gu^\delta + \delta\int\limits_{\gamma} N(v^\delta)\varepsilon(v^\delta) - \delta\int\limits_{\gamma} M(w^\delta)\nabla\nabla w^\delta = 0. \tag{2.10}$$

We can follow the arguments of the paper [1] used for proving the coercivity of the functional E and derive from (2.10) the estimate being uniform in $\delta \geq \delta_0 > 0$,

$$\|(u^\delta, v^\delta, w^\delta)\|_U \leq c. \tag{2.11}$$

In view of (2.11), it can be assumed that as $\delta \to \infty$,

$$(u^\delta, v^\delta, w^\delta) \to (u, v, w) \text{ weakly in } U. \tag{2.12}$$

Moreover, by (2.11), from (2.10) it follows

$$\int_\gamma N(v^\delta)\varepsilon(v^\delta) \le \frac{c_1}{\delta}, \quad -\int_\gamma M(w^\delta)\nabla\nabla w^\delta \le \frac{c_1}{\delta}.$$

Hence, in view of (2.12),

$$\int_\gamma N(v)\varepsilon(v) = 0, \quad \int_\gamma M(w)\nabla\nabla w = 0.$$

This means that

$$\varepsilon_{ij}(v) = 0, \quad w_{,ij} = 0 \text{ on } \gamma, \quad i, j = 1, 2.$$

Consequently,

$$(v_1, v_2, w) \in R(\gamma). \tag{2.13}$$

Introduce the space

$$W = \{u \in H^1(\Omega_\gamma)^3 \mid u|_{\gamma^-} \in R(\gamma)\}$$

and the set

$$Q_\infty = \{u = (u_1, u_2, u_3) \in W \mid [u_3] \ge 0 \text{ on } \gamma\}.$$

In view of (2.12), (2.13), we can pass to the limit in (2.8), (2.9) as $\delta \to \infty$. Take any element $\overline{u} \in Q_\infty$ and denote $(\overline{v}, \overline{w}) = \overline{u}|_{\gamma^-}$, $\overline{v} = (\overline{v}_1, \overline{v}_2)$. Then $(\overline{u}, \overline{v}, \overline{w}) \in Q$. Substituting $(\overline{u}, \overline{v}, \overline{w})$ in (2.9) and passing to the limit as $\delta \to \infty$, we obtain

$$(u, v, w) \in Q_\infty \cap W_1, \tag{2.14}$$

$$\int_{\Omega_\gamma} \sigma(u)\varepsilon(\overline{u} - u) - \int_{\Omega_\gamma} g(\overline{u} - u) \ge 0 \quad \forall (\overline{u}, \overline{v}, \overline{w}) \in Q_\infty, \tag{2.15}$$

where

$$W_1 = \left\{u \in W \mid \int_{\Omega_\gamma} u = 0, \int_{\Omega_\gamma} (u_{i,j} - u_{j,i}) = 0,\ i, j = 1, 2, 3;\ u = (u_1, u_2, u_3)\right\}.$$

Hence, the following statement has been proved.

Theorem 1. *There exists a solution of the problem* (2.14), (2.15) *provided the condition* (1.10) *holds.*

Note that the solution of the problem (2.14), (2.15) is unique. Indeed, assuming an existence of two different solutions u^1, u^2, from (2.14), (2.15) the following relation can be derived for $u = u^1 - u^2$,

$$\int_{\Omega_\gamma} \sigma(u)\varepsilon(u) = 0.$$

Since $u \in W_1$, in view of Proposition 1.4 from [23], this leads to $u = 0$, which is a contradiction.

We can write down the differential formulation of the problem (2.14), (2.15). It is necessary to find a function $u = (u_1, u_2, u_3)$ defined in Ω_γ, as well as a function $\rho_0 \in R(\gamma)$ such that

$$-\operatorname{div} \sigma = g, \quad \sigma = A\varepsilon(u) \text{ in } \Omega_\gamma, \tag{2.16}$$

$$\sigma n = 0 \text{ on } \Gamma, \tag{2.17}$$

$$u^- = \rho_0 \text{ on } \gamma, \tag{2.18}$$

$$[u_3] \geq 0, \quad \sigma_{33}^+ \leq 0, \quad \sigma_{i3}^+ = 0,\ i = 1, 2, \quad [u_3]\sigma_{33}^+ = 0 \text{ on } \gamma, \tag{2.19}$$

$$\int\limits_\gamma [\sigma_{33}]\rho_3 = 0, \quad \int\limits_\gamma (\sigma_{13}^-, \quad \sigma_{23}^-)(\rho_1, \rho_2) = 0 \quad \forall \rho = (\rho_1, \rho_2, \rho_3) \in R(\gamma), \tag{2.20}$$

$$\int\limits_{\Omega_\gamma} u = 0, \quad \int\limits_{\Omega_\gamma} (u_{i,j} - u_{j,i}) = 0, \quad i, j = 1, 2, 3. \tag{2.21}$$

Remark 1. By (2.19), conditions (2.20) can be rewritten in the following equivalent form

$$\int\limits_\gamma [\sigma_{33}]\rho_3 = 0, \quad \int\limits_\gamma ([\sigma_{13}], [\sigma_{23}])(\rho_1, \rho_2) = 0 \quad \forall \rho = (\rho_1, \rho_2, \rho_3) \in R(\gamma), \tag{2.22}$$

or, in the following form

$$\int\limits_\gamma [\sigma m]\rho = 0 \quad \forall \rho \in R(\gamma), \tag{2.23}$$

where $m = (0, 0, 1)$ is a normal vector to γ.

Remark 2. According to (2.20), a principal vector of forces and a principal moment acting on γ are equal to zero.

The following assertion establishes a connection between problems (2.14), (2.15) and (2.16)–(2.21).

Theorem 2. *Problem formulations* (2.14), (2.15) *and* (2.16)–(2.21) *are equivalent provided that the solutions are smooth.*

Proof. Assume that (2.14), (2.15) is fulfilled. It is easy to derive equilibrium equation from (2.16) and boundary condition (2.17). Take test functions in (2.15) in the form $\overline{u} = u \pm \widetilde{u}$, $[\widetilde{u}_3] = 0$ on γ, $\widetilde{u} = (\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3)$, $\widetilde{u}|_{\gamma^-} = (\rho_1, \rho_2, \rho_3) \in R(\gamma)$. It gives

$$\int\limits_{\Omega_\gamma} \sigma(u)\varepsilon(\widetilde{u}) - \int\limits_{\Omega_\gamma} g\widetilde{u} = 0.$$

This relation, by the equilibrium equation, implies

$$\int\limits_\gamma [\sigma(u)m \cdot \widetilde{u}] = 0,$$

or (with $\sigma(u) = \sigma$)

$$\int\limits_{\gamma} [\sigma_{33}]\widetilde{u}_3 + \int\limits_{\gamma} [\sigma_\tau \cdot \widetilde{u}_\tau] = 0; \quad \sigma_\tau = (\sigma_{13}, \sigma_{23}), \quad \widetilde{u}_\tau = (\widetilde{u}_1, \widetilde{u}_2). \tag{2.24}$$

Since $\widetilde{u}_\tau^+$ are arbitrary, the identity (2.24) proves that

$$\sigma_{13}^+ = \sigma_{23}^+ = 0 \text{ on } \gamma. \tag{2.25}$$

By virtue (2.25), the identity (2.24) has the following structure:

$$\int\limits_{\gamma} [\sigma_{33}]\rho_3 + \int\limits_{\gamma} ([\sigma_{13}], [\sigma_{23}])(\rho_1, \rho_2) = 0 \quad \forall \rho = (\rho_1, \rho_2, \rho_3) \in R(\gamma),$$

and we arrive at (2.20) (or (2.22), or (2.23)).

Next, we take in (2.15) test function of the form $\overline{u} = u + \widetilde{u}, \widetilde{u} \in Q_\infty$, $\widetilde{u}_3^+ \geq 0$ on γ; $\operatorname{supp} \widetilde{u} \subset \overline{S}$; $\widetilde{u} = (\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3)$, see Fig. 2. This implies

$$\int\limits_{\Omega_\gamma} \sigma(u)\varepsilon(\widetilde{u}) - \int\limits_{\Omega_\gamma} g\widetilde{u} \geq 0.$$

Consequently,

$$\int\limits_{\gamma} (\sigma_{33}^+ \widetilde{u}_3^+ + \sigma_\tau^+ \cdot \widetilde{u}_\tau^+) \leq 0,$$

and by the choice of the test functions, we obtain

$$\sigma_{33}^+ \leq 0 \text{ on } \gamma.$$

Now, we prove the last relation from (2.19). Assume that $[u_3(y)] > 0$, $y \in \overline{S} \cap \gamma$. Taking test functions in (2.15) as $\overline{u} = u \pm \lambda\varphi$, $\operatorname{supp}\varphi \subset \overline{S}$, λ is small. This provides

$$\int\limits_{\Omega_\gamma} \sigma(u)\varepsilon(\varphi) - \int\limits_{\Omega_\gamma} g\varphi = 0.$$

By the equilibrium equation and (2.25), this identity implies

$$\int\limits_{\gamma} \sigma_{33}^+ \varphi_3 = 0.$$

Hence $\sigma_{33}^+ = 0$ on $\overline{S} \cap \gamma$, what implies the needed equality $\sigma_{33}^+(y)[u_3(y)] = 0$. On the other hand, assuming that $\sigma_{33}^+(y) < 0$, we easy obtain $\sigma_{33}^+(y)[u_3(y)] = 0$.

Thus, all relations (2.16)–(2.21) can be obtained from (2.14), (2.15).

Let us prove the converse. Let (2.16)–(2.21) be fulfilled. We have

$$\int\limits_{\Omega_\gamma} (-\operatorname{div}\sigma - g)(\overline{u} - u) = 0 \quad \forall \overline{u} \in Q_\infty.$$

Integrating by parts, the following relation is derived

$$\int\limits_{\Omega_\gamma} \sigma(u)\varepsilon(\widetilde{u}-u) - \int\limits_{\Omega_\gamma} g(\overline{u}-u) + \int\limits_{\gamma} [\sigma m(\overline{u}-u)] = 0. \tag{2.26}$$

Let us prove that

$$\int\limits_{\gamma} [\sigma m(\overline{u}-u)] \le 0. \tag{2.27}$$

In this case, the inequality (2.15) follows from (2.26). We rewrite (2.27) in the equivalent form

$$\int\limits_{\gamma} [\sigma_{33}(\overline{u}_3-u_3)] + \int\limits_{\gamma} [\sigma_\tau(\overline{u}_\tau-u_\tau)] \le 0. \tag{2.28}$$

Notice that, by the second and the last relations (2.20), the following inequality takes place

$$\int\limits_{\gamma} \sigma_{33}^{+}([\overline{u}_3]-[u_3]) \le 0.$$

Then, taking into account (2.20) and relations $\sigma_{i3}^{+}=0$, $i=1,2$, on γ, we arrive at the conclusion that (2.28) holds. Hence (2.15) is proved. Consequently, the variational inequality (2.14), (2.15) can be derived from (2.16)–(2.21). Theorem 2 is proved.

We can write down one more equivalent formulation of the problem (2.14), (2.15): it is necessary find a function $u=(u_1,u_2,u_3)$ defined in Ω_γ, as well as a function $\rho_0 \in R(\gamma)$ such that

$$-\operatorname{div}\sigma = g, \quad \sigma = A\varepsilon(u) \text{ in } \Omega_\gamma, \tag{2.29}$$

$$\sigma n = 0 \text{ on } \Gamma, \tag{2.30}$$

$$u^- = \rho_0, \quad [u_3] \ge 0 \text{ on } \gamma, \tag{2.31}$$

$$\int\limits_{\gamma} [\sigma m\cdot u] = 0, \quad \int\limits_{\gamma} [\sigma m\cdot \overline{u}] \le 0 \quad \forall \overline{u}\in Q_\infty, \tag{2.32}$$

$$\int\limits_{\Omega_\gamma} u = 0, \quad \int\limits_{\Omega_\gamma} (u_{i,j}-u_{j,i}) = 0, \quad i,j=1,2,3. \tag{2.33}$$

Let us check that problem formulations (2.14), (2.15) and (2.29)–(2.33) are equivalent for smooth solutions. Let (2.14), (2.15) take place. The equilibrium equation from (2.29) and boundary condition (2.30) can be derived by standard arguments. Next, from (2.14), (2.15) it follows

$$\int\limits_{\Omega_\gamma} \sigma(u)\varepsilon(u) - \int\limits_{\Omega_\gamma} gu = 0. \tag{2.34}$$

From here, integrating by parts, we obtain the first relation of (2.32). By (2.34), the inequality (2.15) has the form

$$\int\limits_{\Omega_\gamma} \sigma(u)\varepsilon(\overline{u}) - \int\limits_{\Omega_\gamma} g\overline{u} \geq 0.$$

Hence the second relation from (2.32) follows. Consequently, all relations (2.29)–(2.33) can be obtained from (2.14), (2.15).

Now we prove the converse. Let (2.29)–(2.33) hold. We have

$$\int\limits_{\Omega_\gamma} (-\operatorname{div}\sigma - g)(\overline{u} - u) = 0 \quad \forall \overline{u} \in Q_\infty.$$

Hence

$$\int\limits_{\Omega_\gamma} \sigma(u)\varepsilon(\overline{u} - u) - \int\limits_{\Omega_\gamma} g(\overline{u} - u) + \int\limits_{\gamma} [\sigma m(\overline{u} - u)] = 0. \tag{2.35}$$

By (2.32), we see that

$$\int\limits_{\gamma} [\sigma m(\overline{u} - u)] \leq 0.$$

Thus, from (2.35) the inequality (2.15) follows.

Consequently, we have checked that problem formulation (2.14), (2.15) and (2.29)–(2.33) are equivalent.

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Математическая жизнь

Межгородской научно-исследовательский семинар «Неклассические задачи математической физики»

28 марта 2026 г.

«К 80-летию Тынысбека Шариповича Кальменова. Обзор некоторых научных результатов».

Докладчик: М. Садыбеков (Институт математики и математического моделирование, Алматы, Казахстан).

Дан краткий обзор основных научных результатов профессора Т. Ш. Кальменова. Более подробно изложены критерии корректности задачи Коши для уравнения Лапласа.

11 апреля 2026 г.

«Разрешимость $2d4v$-системы Бродуэлла с периодическими условиями по пространственным переменным».

Докладчик: А. Н. Артюшин (Новосибирский государственный университет, Новосибирск, Россия)

Пусть $\Omega = (0, 2\pi) \times (0, 2\pi)$. В цилиндре $Q = (0, T) \times \Omega$ рассматриваем систему уравнений

$$u_t + u_x = \frac{1}{\varepsilon}(\omega z - uv), \quad v_t - v_x = \frac{1}{\varepsilon}(\omega z - uv),$$

$$\omega_t + \omega_y = \frac{1}{\varepsilon}(uv - \omega z), \quad z_t + z_y = \frac{1}{\varepsilon}(uv - \omega z).$$

Для нее ставится задача Коши с периодическими условиями по (x, y). После масштабирования можно считать, что $\varepsilon = 1$.

Данная система является моделью типа Больцмана. С одной стороны, она наследует характерные особенности уравнения Больцмана. С другой строны, она существенно проще, что дает надежду на возможность ее более содержательного исследования.

В работах Е. В. Радкевича (с соавторами) исследовался вопрос о стабилизации решений к стационарным. Было показано, что имеет место экспоненциальная стабилизация в случае малых гладких регулярных возмущений констант.

Нас будет интересовать самый общий случай.

1. Что будет в случае нерегулярных возмущений? Что можно сказать про предельное стационарное решение? Показано, что даже его гипотетическое

существование связано с разрешимостью одной нетривиальной нелинейной задачи с L_2-проекторами. Показано, что эта задача разрешима при самых общих предположениях.

2. В каких случаях можно доказать существование локального решения? H-теорема Больцмана дает лишь крайне слабую оценку. Уравнения в системе типа Риккати. Поэтому в типичной теореме о локальной разрешимости используется класс ограниченных функций. Оказывается, что на самом деле имеется локальная оценка в классе $L_\infty(0,T;L_2(\Omega))$. Более того, она носит поточечный характер, а не интегральный. Наличие нелинейности осложняет дело, так что такой оценки для доказательства разрешимости не хватает. Тем не менее, при некоторых дополнительных предположениях на начальные данные локальную разрешимость доказать удается.

3. Что можно сказать о регулярных возмущениях самых общих стационарных решений? В более-менее простых случаях удается использовать разложения в ряды Фурье. В общем случае такая возможность не просматривается.

Автор выражает глубокую благодарность Е. В. Радкевичу за внимание к работе и содержательные обсуждения.

25 апреля 2026 г.

«Вариационные неравенства с вырождающимися анизотропными коэффициентами и L^1-данными».

Докладчик: А. А. Ковалевский (Институт математики и механики им. Н. Н. Красовского УрО РАН, Екатеринбург, Институт прикладной математики и механики, Донецк, Россия).

Рассматриваются нелинейные эллиптические вариационные неравенства с вырождающимися (по пространственной переменной) и анизотропными коэффициентами и L^1-правой частью. Разбираются случаи, когда соответствующее множество ограничений принадлежит определенному анизотропному весовому пространству Соболева и некоторому более широкому классу, содержащему несуммируемые функции. Для каждого из этих случаев приводятся результаты о существовании и свойствах решений рассматриваемых вариационных неравенств, а также разнообразные примеры выполнения предположений относительно множества ограничений.

Участники семинара «Неклассические задачи математической физики» от всей души поздравляют профессора Ивана Егоровича Егорова со славным юбилеем — 75-летием! Желаем Вам, Иван Егорович, успехов и счастья в жизни!