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CLASSICAL SOLVABILITY OF THE RADIAL VISCOUS FINGERING PROBLEM IN A HELE—SHAW CELL

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Abstract. We discuss a single-phase radial viscous fingering problem in a Hele–Shaw cell, which is a nonlinear problem with a free boundary for an elliptic equation. Unlike the Stefan problem for heat equation Hele–Shaw problem is of hydrodynamic type. In this paper a single-phase Hele–Shaw problem in a radial flow geometry admits a unique classical solution by applying the same method as for Stefan problem and justifying the vanishing the coefficient of the derivative with respect to time in a parabolic equation.

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1. Introduction

Viscous fingering occurs in the flow of two immiscible, viscous fluids between the plates of a Hele–Shaw cell [1]. Due to pressure gradients or gravity, the initially planar interface separating the two fluids undergoes a Saffman–Taylor instability and develops finger-like structure. The Saffman–Taylor problem [2] is a widely studied example of hydrodynamic pattern formation where interfacial instabilities grow and evolve (see [3–5] and the literatures therein). Experiments and theory focus on two basic flow geometries: (i) rectangular [2] and (ii) radial [6]. For both geometries the initial developments of the interface instability were discussed through the linear stability theory in [3]. However, the significant differences have been reported between the dispersion relation derived from the linear analysis and that obtained from the experiments if the surface tension effect is not negligible, which call attention to the validity of Hele–Shaw approximation and the new formulation with physically reasonable effects. Among them one is the wetting layer effect [7–10] and another is viscous normal stress effect [11, 12].

For the radial fingering phenomena linear stability was discussed in [9] with the former effect and in [12] with the latter effect. Weakly nonlinear analysis was done first by Miranda and Widom [13] without any effect, and recently by Tani in [10] with the wetting effect and in [14] with the viscous normal stress effect.

Most of mathematical results have been established by applying the complex function theory (see [15] and the literatures therein). There are a few results from the exact mathematical analysis: for example, [14, 16] for one-phase problem with surface tension and [17, 18] for two-phase problem in the standard Hölder spaces; [19] for one-phase problem and [20] for one-phase problem with surface tension in the little Hölder spaces; [21–23] for one-phase problem with surface tension in the Sobolev spaces. There are some papers not only [19–22] discussing in the multi-dimensional case, which are absurd from applied viewpoint because Hele–Shaw flows are inherently two-dimensional.

Such a fingering pattern may be found even at a single fluid interface when an incompressible fluid surrounded by air is located between the parallel plates of a horizontal Hele–Shaw cell. We consider this problem in a radial flow geometry and in the classical framework in this paper.

This paper is organized as follows. In section 2 we formulate the problem discussed in this paper and describe the main theorems. In section 3 the problem in section 2 is reformulated. In section 4 the linear problem of the reformulated one is analyzed by following the arguments due to Bazaliĭ [16, 17] and Bizhanova and Solonnikov [24]. The most importance is to get a uniform estimate of the solution with respect to a parameter as in [17, 25]. For that it necessitates to modify the discussion used in the Stefan problem since the boundary conditions on the interface for the Hele–Shaw problem cause further difficulty. In sections 5 and 6 the original nonlinear problem and the passing to the limit of the parameter are discussed.

2. Formulation of the problem

We consider a slow quasi-stationary displacement of air by a fluid in the Hele–Shaw cell. Fluid is assumed to be immiscible and incompressible. The motion is quasi-two-dimensional and all characteristics of the flow are averaged over the cell thickness. This approximation, so-called Hele–Shaw approximation, is traditional for problems of this type.

The motion of such a fluid is described by

$$\nabla \cdot \mathbf{v} = 0, \quad \mathbf{v} = -M\nabla p \quad \text{in } \ \Omega(t),\ t > 0. \tag{2.1}$$

Here $\mathbf{v}$ is the velocity vector field and p is the pressure in the fluid; the second equation in (2.1) means Darcy's law ($M = b^2/(12\mu)$ is mobility; μ is the fluid viscosity; b is the width of two plates). In discussing the fingering problem it is sufficient to consider (2.1) under the following geometrical situation:

$$\Omega(t) = \left\{ x \in \mathbb{R}^2 \mid R_* < |x| < R(t) + \zeta\left(\frac{x}{|x|}, t\right)\right\},$$

where R_* is the radius of the hole through which the displacing fluid is injected at a flow rate $Q(t)$, $R(t)$ is the time-dependent unperturbed radius satisfying

$$\pi R(t)^2 = \pi R_0^2 + \int_0^t Q(\tau)\,\mathrm{d}\tau, \quad R_0 \equiv R(0) > R_*$$

and ζ is the perturbed radius.

In addition, we impose the following boundary conditions:

$$\begin{gathered} \mathbf{v}\cdot\mathbf{n} = \frac{Q(t)}{2\pi R_*} \quad \text{on } \ \Gamma_* = \{x \in \mathbb{R}^2 \mid |x| = R_*\},\ t > 0, \\ \mathbf{v}\cdot\mathbf{n} = V_n, \quad p = p_e \quad \text{on } \ \Gamma(t),\ t > 0, \end{gathered} \tag{2.2}$$

where

$$\Gamma(t) = \left\{ x \in \mathbb{R}^2 \mid |x| = R(t) + \zeta\left(\frac{x}{|x|}, t\right) \right\},$$

V_n is the normal velocity of the interface $\Gamma(t)$, $\mathbf{n}$ is the unit outward normal vector on Γ_* or $\Gamma(t)$ and p_e is the pressure in the displaced air, and the initial conditions are

$$\begin{aligned} \mathbf{v}|_{t=0} = \mathbf{v}^0, \quad p = p^0 \quad \text{on} \;\; \Omega(0) \equiv \Omega, \\ \zeta|_{t=0} = \zeta^0 \quad \text{on} \;\; \Gamma(0) \equiv \Gamma \quad (\zeta^0 > R_* - R_0). \end{aligned} \tag{2.3}$$

Our problem is to find $(\mathbf{v}, p)$ and ζ satisfying (2.1)–(2.3), which is reduced to find p and ζ satisfying

$$\begin{aligned} \Delta p = 0 \quad \text{in } \Omega(t),\; t > 0, \\ -M\nabla p \cdot \mathbf{n} = \frac{Q(t)}{2\pi R_*} \quad \text{on } \Gamma_*,\; t > 0, \\ -M\nabla p \cdot \mathbf{n} = V_n, \quad p = p_e \quad \text{on } \Gamma(t),\; t > 0, \\ p|_{t=0} = p^0 \quad \text{on } \;\Omega, \quad \zeta|_{t=0} = \zeta^0 \quad \text{on } \;\Gamma. \end{aligned} \tag{2.4}$$

As the compatibility conditions p^0 is assumed to satisfy

$$\begin{aligned} \Delta p^0 = 0 \quad \text{in } \;\Omega, \\ -M\nabla p^0 \cdot \mathbf{n} = \frac{Q(0)}{2\pi R_*} \quad \text{on } \;\Gamma_*, \\ p^0 = p_e^0 \equiv p_e|_{t=0} \quad \text{on } \;\Gamma. \end{aligned} \tag{2.5}$$

It is more convenient to rewrite (2.4) in polar coordinates (r, θ):

$$\begin{aligned} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} = 0 \\ (R_* < r < R(t) + \zeta(\theta, t),\; 0 \le \theta < 2\pi,\; t > 0), \\ M\frac{\partial p}{\partial r} = -\frac{Q(t)}{2\pi R_*} \quad (r = R_*,\; 0 \le \theta < 2\pi,\; t > 0), \\ M\left(\frac{\partial p}{\partial r} - \frac{1}{r^2}\frac{\partial \zeta}{\partial \theta}\frac{\partial p}{\partial \theta}\right) = -\frac{\partial}{\partial t}\left(R(t) + \zeta\right), \\ p = p_e \quad (r = R(t) + \zeta(\theta, t),\; 0 \le \theta < 2\pi,\; t > 0), \\ p|_{t=0} = p^0 \quad (R_* < r < R_0 + \zeta^0(\theta),\; 0 \le \theta < 2\pi), \\ \zeta|_{t=0} = \zeta^0 \quad (0 \le \theta < 2\pi). \end{aligned} \tag{2.6}$$

Besides problem (2.6) we consider the following problem:

$$
\begin{gathered}
\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 p}{\partial\theta^2}=\varepsilon\frac{\partial p}{\partial t}+\varepsilon f\\
(R_*<r<R(t)+\zeta(\theta,t),\ 0\le\theta<2\pi,\ t>0),\\
M\frac{\partial p}{\partial r}=-\frac{Q(t)}{2\pi R_*}\quad (r=R_*,\ 0\le\theta<2\pi,\ t>0),\\
M\left(\frac{\partial p}{\partial r}-\frac{1}{r^2}\frac{\partial\zeta}{\partial\theta}\frac{\partial p}{\partial\theta}\right)=-\frac{\partial}{\partial t}\left(R(t)+\zeta\right),\\
p=p_e\quad (r=R(t)+\zeta(\theta,t),\ 0\le\theta<2\pi,\ t>0),\\
p|_{t=0}=p^0\quad (R_*<r<R_0+\zeta^0(\theta),\ 0\le\theta<2\pi),\\
\zeta\,\big|_{t=0}=\zeta^0\quad (0\le\theta<2\pi)
\end{gathered}
\tag{2.7}
$$

with $\varepsilon>0$ and f being given later.

Now let us transform the free boundary problem (2.7) into the problem on a fixed domain. Introduce the transformation from

$$\Omega(t)=\{R_*<r<R(t)+\zeta(\theta,t),\ 0\le\theta<2\pi\}$$

onto

$$\Omega=\{R_*<r'<R_0+\zeta^0(\theta'),\ 0\le\theta'<2\pi\}$$

by the change of the independent variables

$$r'=\frac{R_0+\zeta^0-R_*}{R+\zeta-R_*}(r-R_*)+R_*,\quad \theta'=\theta,\quad t'=t,$$

and the unknown functions

$$p(r,\theta,t)=p'(r',\theta',t'),\quad \zeta(\theta,t)=\zeta'(\theta',t').$$

By omitting all the *primes* for simplicity, problem (2.7) takes the form

$$
\begin{gathered}
\varepsilon\frac{\partial p}{\partial t}=\mathscr{L}^{\varepsilon}_{\zeta}p-\varepsilon f\quad \text{in}\ \ \Omega,\ t>0,\\
\frac{\partial p}{\partial r}=-\frac{Q(t)}{2\pi R_* M}\frac{R+\zeta-R_*}{R_0+\zeta^0-R_*}\quad \text{on}\ \ \Gamma_*\equiv\{r=R_*,\ \theta\in[0,2\pi]\},\ t>0,\\
\frac{\partial\zeta}{\partial t}-b_2(\zeta)\frac{\partial p}{\partial r}-b_1(\zeta)\frac{\partial p}{\partial\theta}=-\frac{Q(t)}{2\pi R},\\
p=p_e\quad \text{on}\ \ \Gamma\equiv\{r=R_0+\zeta^0(\theta),\ \theta\in[0,2\pi]\},\ t>0,\\
p|_{t=0}=p^0\quad \text{on}\ \ \Omega,\\
\zeta|_{t=0}=\zeta^0\quad \text{on}\ \ [0,2\pi].
\end{gathered}
\tag{2.8}
$$

Here

$$
\mathscr{L}^{\varepsilon}_{\zeta}\equiv\mathscr{L}^{\varepsilon}_{\zeta}\left(r,\theta;\frac{\partial}{\partial r},\frac{\partial}{\partial\theta}\right)=\frac{1}{\left(R_*+\frac{R+\zeta-R_*}{R_0+\zeta^0-R_*}(r-R_*)\right)^2}
$$
$$
\times\left[\frac{\partial^2}{\partial\theta^2}+2\left(\frac{1}{R_0+\zeta^0-R_*}\frac{\mathrm{d}\zeta^0}{\mathrm{d}\theta}-\frac{1}{R+\zeta-R_*}\frac{\partial\zeta}{\partial\theta}\right)(r-R_*)\frac{\partial^2}{\partial r\partial\theta}
$$

$$+\left(\left(R_*+\frac{R+\zeta-R_*}{R_0+\zeta^0-R_*}(r-R_*)\right)^2\left(\frac{R_0+\zeta^0-R_*}{R+\zeta-R_*}\right)^2\right.$$
$$\left.+\left(\frac{1}{R_0+\zeta^0-R_*}\frac{\mathrm{d}\zeta^0}{\mathrm{d}\theta}-\frac{1}{R+\zeta-R_*}\frac{\partial\zeta}{\partial\theta}\right)^2(r-R_*)^2\right)\frac{\partial^2}{\partial r^2}\Bigg]$$
$$+\left\{\varepsilon\frac{r-R_*}{R+\zeta-R_*}\frac{\partial}{\partial t}(R+\zeta)+\frac{1}{R_*+\frac{R+\zeta-R_*}{R_0+\zeta^0-R_*}(r-R_*)}\frac{R_0+\zeta^0-R_*}{R+\zeta-R_*}\right.$$
$$+\frac{r-R_*}{\left(R_*+\frac{R+\zeta-R_*}{R_0+\zeta^0-R_*}(r-R_*)\right)^2}\left[\frac{\partial}{\partial\theta}\left(\frac{1}{R_0+\zeta^0-R_*}\frac{\mathrm{d}\zeta^0}{\mathrm{d}\theta}-\frac{1}{R+\zeta-R_*}\frac{\partial\zeta}{\partial\theta}\right)\right.$$
$$\left.\left.+\left(\frac{1}{R_0+\zeta^0-R_*}\frac{\mathrm{d}\zeta^0}{\mathrm{d}\theta}-\frac{1}{R+\zeta-R_*}\frac{\partial\zeta}{\partial\theta}\right)^2\right]\right\}\frac{\partial}{\partial r};$$
$$b_2(\zeta)=M\left[\frac{R_0+\zeta^0-R_*}{R+\zeta-R_*}\left(1+\frac{1}{(R_0+\zeta^0)^2}\left(\frac{\partial\zeta}{\partial\theta}\right)^2\right)-\frac{1}{(R_0+\zeta^0)^2}\frac{\partial\zeta}{\partial\theta}\frac{\mathrm{d}\zeta^0}{\mathrm{d}\theta}\right],$$
$$b_1(\zeta)=-M\frac{1}{(R_0+\zeta^0)^2}\frac{\partial\zeta}{\partial\theta}.$$

Now we choose

$$f=\frac{r-R_*}{R_0+\zeta^0-R_*}\frac{\partial p^0}{\partial r}\frac{\partial}{\partial t}(R+\zeta)\bigg|_{t=0}\quad\text{on }\ \Omega.$$

Before describing the main results, we introduce function spaces. For a domain Ω in $\mathbb{R}^n$ ($n\in\mathbb{N}$) and any $T>0$ let $C^l(\overline{\Omega})$ and $C^{l,l/2}_{x,t}(\overline{Q}_T)$ ($\overline{Q}_T\equiv\overline{\Omega}\times[0,T]$) with $l=k+\alpha$, $k\in\mathbb{Z}$, $k\geq 0$, $\alpha\in(0,1)$ be the standard Hölder spaces constructed with the use of the following semi-norms:

$$|u|^{(\alpha)}=|u|^{(0)}+\langle u\rangle^{(\alpha)},\quad |u|^{(0)}=\sup_{(x,t)\in\overline{Q}_T}|u(x,t)|,\quad \langle u\rangle^{(\alpha)}=\langle u\rangle^{(\alpha)}_x+\langle u\rangle^{(\alpha/2)}_t,$$
$$\langle u\rangle^{(\alpha)}_x\equiv\sup_{x,y\in\overline{\Omega},\,t\in[0,T]}\frac{|u(x,t)-u(y,t)|}{|x-y|^\alpha},\quad \langle u\rangle^{(\alpha)}_t\equiv\sup_{x\in\overline{\Omega},\,t,t'\in[0,T]}\frac{|u(x,t)-u(x,t')|}{|t-t'|^\alpha}$$

(see [26]). We also use the semi-norm

$$[u]^{(\alpha,\beta)}\equiv\sup_{x,y\in\overline{\Omega}\,t,t'\in[0,T]}\frac{|u(x,t)-u(y,t)-u(x,t')+u(y,t')|}{|x-y|^\alpha|t-t'|^\beta},\quad \alpha,\beta\in(0,1),$$

and define the Banach spaces $E^{k+\alpha}(\overline{Q}_T)$ ($k=0,1,2$) that are obtained as the completion of infinitely differential functions in respective norms

$$\|u\|_\alpha=\|u\|_{E^\alpha(\overline{Q}_T)}\equiv E^{\alpha,\alpha/2}[u]=\sup_{(x,t)\in\overline{Q}_T}|u(x,t)|+\langle u\rangle^{(\alpha)}+[u]^{(\alpha,\alpha/2)},$$
$$D^{\alpha,\alpha}[u]=\sup_{(x,t)\in\overline{Q}_T}|u(x,t)|+\langle u\rangle^{(\alpha)}_x+\langle u\rangle^{(\alpha)}_t+[u]^{(\alpha,\alpha)},$$
$$\|u\|_{k+\alpha}=\|u\|_{E^{k+\alpha}(\overline{Q}_T)}=E^{\alpha,\alpha/2}\left[\mathrm{D}^k_x u\right]+\sum_{j=0}^{k-1}D^{\alpha,\alpha}\left[\mathrm{D}^j_x u\right]$$

$$\Big(\mathrm{D}_x^k = \sum_{|j|=k} \partial_x^j \quad (j \text{ is a multi-index}), \quad k=1,2\Big),$$

and the space $P_\varepsilon^{2+\alpha}(\overline{Q}_T)$ with the norm $\|u\|_{P_\varepsilon^{2+\alpha}(\overline{Q}_T)} = \|\varepsilon\partial_t u\|_\alpha + \|u\|_{2+\alpha}$. We also introduce the spaces

$$\widehat{E}^{2+\alpha}(\overline{Q}_T) = \{u \mid u \in E^{2+\alpha}(\overline{Q}_T),\ \partial_t u \in E^{1+\alpha}(\overline{Q}_T)\},$$

$$\|u\|_{\widehat{E}^{2+\alpha}(\overline{Q}_T)} = \|u\|_{2+\alpha} + \|\partial_t u\|_{1+\alpha},$$

$$\check{E}^{2+\alpha}(\overline{Q}_T) = \left\{u \mid u \in E^{2+\alpha}(\overline{Q}_T),\ \partial_t u \in E^{\alpha}(\overline{Q}_T)\right\},$$

$$\|u\|_{\check{E}^{2+\alpha}(\overline{Q}_T)} = \|u\|_{2+\alpha} + \|\partial_t u\|_{\alpha},$$

$$E_*^{2+\alpha}(\overline{Q}_T) = \left\{u \mid \|u\|_{E_*^{2+\alpha}(\overline{Q}_T)} \equiv \sum_{k=0}^{2} E^{\alpha,\alpha/2}\left[\mathrm{D}_x^k u\right] < \infty\right\},$$

$$\widehat{E}_*^{2+\alpha}(\overline{Q}_T) = \left\{u \mid \|u\|_{\widehat{E}_*^{2+\alpha}(\overline{Q}_T)} \equiv \sum_{k=0}^{2} E^{\alpha,\alpha/2}\left[\mathrm{D}_x^k u\right] + \sum_{k=0}^{1} E^{\alpha,\alpha/2}[\mathrm{D}_x^k \partial_t u] < \infty\right\},$$

$$\check{E}_*^{2+\alpha}(\overline{Q}_T) = \left\{u \mid \|u\|_{\check{E}_*^{2+\alpha}(\overline{Q}_T)} \equiv \sum_{k=0}^{2} E^{\alpha,\alpha/2}\left[\mathrm{D}_x^k u\right] + E^{\alpha,\alpha/2}[\partial_t u] < \infty\right\}.$$

Denote by $\underset{0}{E}^{k+\alpha}(\overline{Q}_T)$, $\underset{0}{P_\varepsilon}^{2+\alpha}(\overline{Q}_T)$, etc. the spaces of the corresponding spaces whose elements are equal to zero at $t=0$ together with their admissible derivatives with respect to t.

For a smooth manifold Γ in $\mathbb{R}^n$ $\quad C^l(\Gamma)$, $C_{x,t}^{l,l/2}(\Gamma_T)$, etc. are defined with the help of partition of unity and of local maps.

The following is our main result.

Theorem 2.1. *Let $T>0$ and $\alpha \in (0,1)$. Assume that $(p^0,\zeta^0) \in C^{3+\alpha}(\overline{\Omega}) \times C^{4+\alpha}([0,2\pi])$ satisfy the compatibility conditions, $\partial p^0/\partial r < 0$ on Γ, $p_e \in C_{\theta,t}^{3+\alpha,(3+\alpha)/2}(\Gamma_T)$ and $Q \in C^\alpha([0,T])$. Then there exists $T_0^* > 0$ depending on the data of the problem such that problem (2.8) with $\varepsilon = 0$ has a unique solution $(p,\zeta) \in E^{2+\alpha}(\overline{Q}_{T_0^*}) \times \widehat{E}^{2+\alpha}(\Gamma_{T_0^*})$.*

Theorem 2.1 is proved in several steps. First we consider problem (2.8) with $\varepsilon > 0$ in the classes of smooth functions, and then show that this solution converges as $\varepsilon \to 0$. For that it is necessary to use the function classes where the uniform estimate in ε holds.

Theorem 2.2. *Let $\varepsilon_0 > 0$ and the corresponding assumptions in Theorem 2.1 hold. Then there exists $T_0 > 0$ depending on the data of the problem and ε_0 such that problem (2.8) with a fixed $\varepsilon \in (0,\varepsilon_0]$ has a unique solution $(p_\varepsilon,\zeta_\varepsilon) \in P_\varepsilon^{2+\alpha}(\overline{Q}_{T_0}) \times \widehat{E}^{2+\alpha}(\Gamma_{T_0})$.*

3. Reformulation of the problem

Let $\bar\zeta \in C^{4+\alpha,2+\alpha/2}_{\theta,t}([0,2\pi]\times[0,T])$ be an extension of ζ^0 such that

$$\left(\bar\zeta, \frac{\partial\bar\zeta}{\partial t}, \frac{\partial^2\bar\zeta}{\partial t^2}\right)\bigg|_{t=0} = \left(\zeta^0, \frac{\partial\zeta}{\partial t}, \frac{\partial^2\zeta}{\partial t^2}\right)\bigg|_{t=0},$$

where $(\partial\zeta/\partial t, \partial^2\zeta/\partial t^2)|_{t=0}$ are obtained from the third equation in (2.8) and its derivative in t at $t=0$.

We seek a solution of problem (2.8) in the form

$$p = p^* + p^0 + \frac{r-R_*}{R+\bar\zeta-R_*}\frac{\partial p^0}{\partial r}\zeta^*, \quad \zeta = \zeta^* + \bar\zeta. \tag{3.1}$$

Then (2.8) becomes

$$\begin{gathered}
\varepsilon\frac{\partial p^*}{\partial t} = \mathscr{L}_* p^* + \Phi^\varepsilon \quad \text{in } \ \Omega,\ t>0,\\
\frac{\partial p^*}{\partial r} = \Psi_* \quad \text{on } \ \Gamma_*,\ t>0,\\
\frac{\partial\zeta^*}{\partial t} - b_2(\bar\zeta)\frac{\partial p^*}{\partial r} - b_1(\bar\zeta)\frac{\partial p^*}{\partial\theta} = \Psi_1, \quad p^* - d(\bar\zeta)\zeta^* = \Psi_2 \text{ on } \ \Gamma,\ t>0,\\
p^*|_{t=0} = 0 \text{ on } \ \Omega, \quad \zeta^*\ \big|_{t=0} = 0 \text{ on } \ [0,2\pi].
\end{gathered} \tag{3.2}$$

Here

$$\begin{aligned}
\mathscr{L}_* \equiv \mathscr{L}_*\left(r,\theta;\frac{\partial}{\partial r},\frac{\partial}{\partial\theta}\right) &= \frac{1}{\left(R_* + \frac{R+\bar\zeta-R_*}{R_0+\zeta^0-R_*}(r-R_*)\right)^2}\Bigg[\frac{\partial^2}{\partial\theta^2}\\
&+ 2\left(\frac{1}{R_0+\zeta^0-R_*}\frac{\mathrm{d}\zeta^0}{\mathrm{d}\theta} - \frac{1}{R+\bar\zeta-R_*}\frac{\partial\bar\zeta}{\partial\theta}\right)(r-R_*)\frac{\partial^2}{\partial r\partial\theta}\\
&+ \left(\left(R_* + \frac{R+\bar\zeta-R_*}{R_0+\zeta^0-R_*}(r-R_*)\right)^2\left(\frac{R_0+\zeta^0-R_*}{R+\bar\zeta-R_*}\right)^2\right.\\
&\left.+\left(\frac{1}{R_0+\zeta^0-R_*}\frac{\mathrm{d}\zeta^0}{\mathrm{d}\theta} - \frac{1}{R+\bar\zeta-R_*}\frac{\partial\bar\zeta}{\partial\theta}\right)^2(r-R_*)^2\right)\frac{\partial^2}{\partial r^2}\Bigg];
\end{aligned}$$

$$\Phi^\varepsilon = \Phi^\varepsilon(p^*,\zeta^*) = \mathscr{L}^\varepsilon_\zeta p - \mathscr{L}_* p^* - \varepsilon(r-R_*)\frac{\partial p^0}{\partial r}\frac{\partial}{\partial t}\left(\frac{\zeta^*}{R+\bar\zeta-R_*}\right) - \varepsilon f;$$

$$\Psi_* = \Psi_*(\zeta^*) = -\frac{\partial}{\partial r}\left(p^0 + \frac{r-R_*}{R+\bar\zeta-R_*}\frac{\partial p^0}{\partial r}\right)\zeta^* - \frac{R+\zeta-R_*}{R_0+\zeta^0-R_*}\frac{Q(t)}{2\pi R_* M},$$

$$\Psi_1 = \Psi_1(p^*,\zeta^*) = b_2(\zeta)\frac{\partial p}{\partial r} + b_1(\zeta)\frac{\partial p}{\partial\theta} - b_2(\bar\zeta)\frac{\partial p^*}{\partial r} - b_1(\bar\zeta)\frac{\partial p^*}{\partial\theta} - \frac{\partial\bar\zeta}{\partial t} - \frac{Q}{2\pi R},$$

$$\Psi_2 = p_e - p^0;$$

$$d(\bar\zeta) = -\frac{R_0+\zeta^0-R_*}{R+\bar\zeta-R_*}\frac{\partial p^0}{\partial r}$$

with (p,ζ) substituted by (3.1). Note that the assumption $\partial p^0/\partial r < 0$ on Γ implies $d(\bar\zeta) > 0$.

4. Linear Problem

In this section we consider the following linear problem:

$$
\begin{gathered}
\varepsilon \frac{\partial u}{\partial t} - \mathscr{L}_* u = \phi \quad \text{in} \ \ \Omega, \ t > 0, \\
\frac{\partial u}{\partial r} = \psi_* \quad \text{on} \ \ \Gamma_*, \ t > 0, \\
\frac{\partial \varrho}{\partial t} - b_2(\bar{\zeta}) \frac{\partial u}{\partial r} - b_1(\bar{\zeta}) \frac{\partial u}{\partial \theta} = \psi_1, \quad u - d(\bar{\zeta}) \varrho = \psi_2 \quad \text{on} \ \ \Gamma, \ t > 0, \\
u|_{t=0} = 0 \quad \text{on} \ \ \Omega, \quad \varrho|_{t=0} = 0 \quad \text{on} \ \ [0, 2\pi]
\end{gathered} \tag{4.1}
$$

for given ϕ, ψ_*, ψ_1, ψ_2 under the conditions $b_2 > 0$, $d > 0$.

First we study three model problems in the whole- and half-spaces:

$$
\varepsilon \frac{\partial u}{\partial t} - \mathscr{L} u = f \ ((x_1, x_2) \in \mathbb{R}^2, \ t > 0), \quad u|_{t=0} = 0; \tag{4.2}
$$

$$
\begin{gathered}
\varepsilon \frac{\partial u}{\partial t} - \mathscr{L} u = f \ \left((x_1, x_2) \in \mathbb{R}^2_+ \equiv \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 > 0\}, \ t > 0\right), \\
\left.\frac{\partial u}{\partial x_2}\right|_{x_2 = 0} = 0, \quad u|_{t=0} = 0;
\end{gathered} \tag{4.3}
$$

$$
\begin{gathered}
\varepsilon \frac{\partial u}{\partial t} - \mathscr{L} u = 0 \quad \left((x_1, x_2) \in \mathbb{R}^2_+, \ t > 0\right), \\
\frac{\partial \varrho}{\partial t} - b \left.\frac{\partial u}{\partial x_2}\right|_{x_2 = 0} = g_1, \quad u - d\varrho|_{x_2 = 0} = g_2, \\
(u, \varrho)|_{t=0} = 0.
\end{gathered} \tag{4.4}
$$

In the above

$$
\mathscr{L} = \sum_{j,k=1}^{2} a_{jk} \frac{\partial^2}{\partial x_j \partial x_k}
$$

is the second order partial differential operator with positive real coefficients a_{jk} which constitute positive definite symmetric matrix, and b and d are positive constants.

For problems (4.2)–(4.4) we can assume without loss of generality $a_{jk} = \delta_{jk}$ by changing the independent variables (cf. [26]). Then the solutions to problems (4.2) and (4.3) for $\mathscr{L} = \Delta$ are given by

$$
u(x, t) = \int_0^t \mathrm{d}\tau \int_{\mathbb{R}^2} \Gamma_\varepsilon(x - y, t - \tau) f(y, \tau) \, \mathrm{d}y, \tag{4.5}
$$

and

$$
u(x, t) = \int_0^t \mathrm{d}\tau \int_{\mathbb{R}^2} N_\varepsilon(x - y, t - \tau) f(y, \tau) \, \mathrm{d}y, \tag{4.6}
$$

respectively, where

$$
\Gamma_\varepsilon(x, t) = \varepsilon^{n/2 - 1} (4\pi t)^{-n/2} \exp\left[-\frac{\varepsilon |x|^2}{4t}\right], \quad x = (x', x_n),
$$

$$N_\varepsilon(x,t) = \Gamma_\varepsilon(x', x_n, t) + \Gamma_\varepsilon(x', -x_n, t), \quad n = 2.$$

Applying the same arguments as in the estimation of the volume potential for the heat equation to the integrals in (4.5) and (4.6), we have uniform estimates with respect to ε for the solutions of the problems (4.2) and (4.3) (see, [17, 26])

$$\varepsilon \left\| \frac{\partial u}{\partial t} \right\|_\alpha + \|u\|_{2+\alpha} \le C_1 \|f\|_\alpha. \tag{4.7}$$

For problem (4.4) by making use of the Fourier–Laplace transformation

$$\mathscr{F}\mathscr{L}[u](\xi, x_2, s) \equiv \tilde{u}(\xi, x_2, s) = \int_0^\infty \mathrm{e}^{-st}\,\mathrm{d}t \int_{-\infty}^\infty \mathrm{e}^{-\mathrm{i}\,\xi x_1} u(x_1, x_2, t)\,\mathrm{d}x_1$$

as in [16, 17, 24], the parabolic equation is reduced to an ordinary differential equation in x_2, so that we get

$$\tilde{u}(\xi, x_2, s) = \tilde{v}(\xi, s)\,\mathrm{e}^{-r_\varepsilon x_2} \quad (x_2 > 0), \quad r_\varepsilon = r_\varepsilon(s, \xi) = \sqrt{\varepsilon s + \xi^2}, \quad \operatorname{Re} r_\varepsilon > 0. \tag{4.8}$$

From the boundary conditions in (4.4) it follows that

$$b r_\varepsilon \tilde{v} + s\tilde{\varrho} = \tilde{g}_1, \quad \tilde{v} - d\tilde{\varrho} = \tilde{g}_2.$$

Solving these, we obtain

$$\tilde{\varrho} = \frac{1}{s + dbr_\varepsilon}(\tilde{g}_1 - br_\varepsilon \tilde{g}_2), \quad \tilde{v} = d\tilde{\varrho} + \tilde{g}_2. \tag{4.9}$$

The solution of problem (4.4) is given through the inverse Fourier–Laplace transformation

$$v(x_1, t) \equiv (\mathscr{F}\mathscr{L})^{-1}[\tilde{v}] = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}x_1\xi}\,\mathrm{d}\xi \int_{\operatorname{Re}s = a > 0} \mathrm{e}^{st} \tilde{v}(\xi, s)\,\mathrm{d}s$$

as follows.

$$\begin{gathered} \varrho(x_1, t) = (\mathscr{F}\mathscr{L})^{-1}\left[\frac{1}{s + dbr_\varepsilon}\right] * (g_1 - b(\mathscr{F}\mathscr{L})^{-1}[r_\varepsilon \tilde{g}_2]), \\ v(x_1, t) = d\varrho + g_2, \end{gathered} \tag{4.10}$$

where $*$ means a convolution with respect to x_1 and t.

By following the arguments in [16], let us derive the explicit representation of

$$Z_\varepsilon(x_1, t) \equiv (\mathscr{F}\mathscr{L})^{-1}\left[\frac{1}{s + dbr_\varepsilon(s, \xi)}\right], \quad t > 0. \tag{4.11}$$

For simplicity let $db = 1$. Since

$$(s + \sqrt{\varepsilon s + \xi^2})^{-1} = \int_0^\infty \exp[-\tau(s + \sqrt{\varepsilon s + \xi^2})]\,\mathrm{d}\tau,$$

$$\frac{1}{2\pi\mathrm{i}}\int\limits_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty}\exp[-\tau(s+\sqrt{\varepsilon s+\xi^2})+st]\,\mathrm{d}s$$
$$=\frac{1}{2\pi\mathrm{i}}\int\limits_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty}\exp[-\tau s+st]\,\mathrm{d}s \underset{t}{*} \frac{1}{2\pi\mathrm{i}}\int\limits_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty}\exp[-\tau\sqrt{\varepsilon s+\xi^2}+st]\,\mathrm{d}s$$
$$=\delta(t-\tau)\underset{t}{*}\frac{\sqrt{\varepsilon}}{2\sqrt{\pi}}\tau t^{-3/2}\exp\left[-\frac{\xi^2 t}{\varepsilon}-\frac{\varepsilon\tau^2}{4t}\right]\quad (a>0),$$

where $\underset{t}{*}$ means a convolution with respect to t, so that

$$\widetilde{Z}_\varepsilon(\xi,t)=\int\limits_0^\infty\delta(t-\tau)\underset{t}{*}\frac{\sqrt{\varepsilon}}{2\sqrt{\pi}}\tau t^{-3/2}\exp\left[-\frac{\xi^2 t}{\varepsilon}-\frac{\varepsilon\tau^2}{4t}\right]\mathrm{d}\tau$$
$$=\frac{\sqrt{\varepsilon}}{2\sqrt{\pi}}\int\limits_0^t\tau(t-\tau)^{-3/2}\exp\left[-\frac{\xi^2}{\varepsilon}(t-\tau)-\frac{\varepsilon\tau^2}{4(t-\tau)}\right]\mathrm{d}\tau$$
$$=\frac{1}{2\sqrt{\pi}}\int\limits_0^{t/\varepsilon}\frac{t-\varepsilon z}{z^{3/2}}\exp\left[-\xi^2 z-\frac{(t-\varepsilon z)^2}{4z}\right]\mathrm{d}z.$$

Therefore, we have

$$Z_\varepsilon(x_1,t)=\frac{1}{2\pi}\int\limits_0^\infty\widetilde{Z}_\varepsilon(\xi,t)\cos(\xi x_1)\,\mathrm{d}\xi$$
$$=\frac{1}{4\pi}\int\limits_0^{t/\varepsilon}\frac{t-\varepsilon z}{z^2}\exp\left[-\frac{x_1^2}{4z}-\frac{(t-\varepsilon z)^2}{4z}\right]\mathrm{d}z. \quad (4.12)$$

From (4.12) it easily follows

Lemma 4.1. *Inequalities*

$$\begin{gathered}|Z_\varepsilon(x_1,t)|\le C_2 t\frac{1+\sqrt{\varepsilon t}}{x_1^2+t^2},\\ \left|\frac{\partial}{\partial t}Z_\varepsilon(x_1,t)\right|+\left|\frac{\partial}{\partial x_1}Z_\varepsilon(x_1,t)\right|\le C_2\frac{1+\sqrt{\varepsilon t}}{x_1^2+t^2},\\ \left|\frac{\partial^2}{\partial t\partial x_1}Z_\varepsilon(x_1,t)\right|+\left|\frac{\partial^2}{\partial x_1^2}Z_\varepsilon(x_1,t)\right|\le C_2\frac{1+\varepsilon t}{(x_1^2+t^2)^{3/2}}\end{gathered}\quad (4.13)$$

hold with a constant C_2 independent of ε.

Lemma 4.1 implies

$$\lim_{t\to 0}\int\limits_{-\infty}^{\infty}Z_\varepsilon(x_1-\xi,t)f(\xi)\,\mathrm{d}\xi=f(x_1) \quad (4.14)$$

for any bounded continuous function $f(x_1)$.

Introduce the notation

$$w(x_1,t) = (Z_\varepsilon * g)(x_1,t) = \int_0^t \mathrm{d}\tau \int_{-\infty}^{\infty} Z_\varepsilon(x_1-y,t-\tau)g(y,\tau)\,\mathrm{d}y,$$

$$w_h(x_1,t) = \int_0^{t-h} \mathrm{d}\tau \int_{-\infty}^{\infty} Z_\varepsilon(x_1-y,t-\tau)g(y,\tau)\,\mathrm{d}y \quad (h>0).$$

For w_h it is clear to hold

$$\frac{\partial}{\partial t}w_h(x_1,t) = \int_0^{t-h} \mathrm{d}\tau \int_{-\infty}^{\infty} \frac{\partial}{\partial t}Z_\varepsilon(x_1-y,t-\tau)(g(y,\tau)-g(x_1,\tau))\,\mathrm{d}y$$
$$+\int_0^{t-h} g(x_1,\tau)\,\mathrm{d}\tau \int_{-\infty}^{\infty} \frac{\partial}{\partial t}Z_\varepsilon(x_1-y,t-\tau)\,\mathrm{d}y + \int_{-\infty}^{\infty} Z_\varepsilon(x_1-y,h)g(y,t-h)\,\mathrm{d}y.$$

Making use of (4.14) and the formula

$$\frac{\partial}{\partial t}\widetilde{Z}_\varepsilon(\xi,t) = -\frac{1}{\sqrt{\pi}}\int_0^{t/\varepsilon} \frac{\varepsilon(t-\varepsilon z)}{z^{1/2}(t+\varepsilon z)^2}\exp\left[-\xi^2 z - \frac{(t-\varepsilon z)^2}{4z}\right]\mathrm{d}z$$
$$-\frac{1}{\sqrt{\pi}}\int_0^{t/\varepsilon} \frac{\xi^2(t-\varepsilon z)}{z^{1/2}(t+\varepsilon z)}\exp\left[-\xi^2 z - \frac{(t-\varepsilon z)^2}{4z}\right]\mathrm{d}z \quad (4.15)$$

together with the estimates in Lemma 4.1, we have after passing to the limit $h \to 0$

$$\frac{\partial}{\partial t}w(x_1,t) = \int_0^{t} \mathrm{d}\tau \int_{-\infty}^{\infty} \frac{\partial}{\partial t}Z_\varepsilon(x_1-y,t-\tau)(g(y,\tau)-g(x_1,\tau))\,\mathrm{d}y$$
$$+\int_0^{t} g(x_1,\tau)\,\mathrm{d}\tau \int_{-\infty}^{\infty} \frac{\partial}{\partial t}Z_\varepsilon(x_1-y,t-\tau)\,\mathrm{d}y + g(x_1,t). \quad (4.16)$$

Analogously we have

$$\frac{\partial^2}{\partial x_1^2}w(x_1,t) = \int_0^{t} \mathrm{d}\tau \int_{-\infty}^{\infty} \frac{\partial}{\partial x_1}Z_\varepsilon(x_1-y,t-\tau)\left(\frac{\partial}{\partial y}g(y,\tau) - \frac{\partial}{\partial x_1}g(x_1,\tau)\right)\mathrm{d}y. \quad (4.17)$$

We begin by estimating the first term denoted by $w^\dagger$ in the right hand side of (4.16). Following the arguments in [26], we get

$$w^\dagger(x_1,t) - w^\dagger(x_1',t)$$
$$= \int_0^{t} \mathrm{d}\tau \int_{|x_1-y|\le 2|x_1-x_1'|} \frac{\partial}{\partial t}Z_\varepsilon(x_1-y,t-\tau)(g(y,\tau)-g(x_1,\tau))\,\mathrm{d}y$$

$$
\begin{aligned}
&- \int_0^t \mathrm{d}\tau \int_{|x_1-y|\le 2|x_1-x_1'|} \frac{\partial}{\partial t} Z_\varepsilon(x_1'-y,t-\tau)(g(y,\tau)-g(x_1',\tau))\,\mathrm{d}y \\
&+ \int_0^t \mathrm{d}\tau \int_{|x_1-y|\ge 2|x_1-x_1'|} \left(\frac{\partial}{\partial t} Z_\varepsilon(x_1-y,t-\tau) - \frac{\partial}{\partial t} Z_\varepsilon(x_1'-y,t-\tau) \right) \\
&\qquad \times (g(y,\tau)-g(x_1,\tau))\,\mathrm{d}y \\
&+ \int_0^t (g(x_1',\tau)-g(x_1,\tau))\,\mathrm{d}\tau \int_{|x_1-y|\ge 2|x_1-x_1'|} \frac{\partial}{\partial t} Z_\varepsilon(x_1'-y,t-\tau)\,\mathrm{d}y \equiv \sum_{j=1}^4 I_j. \qquad (4.18)
\end{aligned}
$$

Applying Lemma 4.1 to I_1, we have

$$
\begin{aligned}
|I_1| &\le C_3''\langle g\rangle_x^{(\alpha)} \int_{|x_1-y|\le 2|x_1-x_1'|} |x_1-y|^\alpha\,\mathrm{d}y \int_0^t \frac{1+\sqrt{\varepsilon(t-\tau)}}{|x_1-y|^2+(t-\tau)^2}\,\mathrm{d}\tau \\
&\le C_3'(1+\sqrt{\varepsilon t})\langle g\rangle_x^{(\alpha)} \int_{|x_1-y|\le 2|x_1-x_1'|} |x_1-y|^{\alpha-1}\,\mathrm{d}\tau \\
&\le C_3(\varepsilon_0,T)\langle g\rangle_x^{(\alpha)}|x_1-x_1'|^\alpha.
\end{aligned}
$$

I_2 is estimated by just the same way as I_1. For I_3 we do in the same way with the help of the mean value theorem. Finally for I_4 we get

$$
\begin{aligned}
|I_4| &\le C_4'\langle g\rangle_x^{(\alpha)}|x_1-x_1'|^\alpha \int_0^t \mathrm{d}\tau \left| \int_{|x_1-y|\ge 2|x_1-x_1'|} \frac{\partial}{\partial t} Z_\varepsilon(x_1'-y,\tau)\,\mathrm{d}y \right| \\
&\le C_4(\varepsilon_0,T)\langle g\rangle_x^{(\alpha)}|x_1-x_1'|^\alpha
\end{aligned}
$$

due to (4.15) and Lemma 4.1. Second term in the right hand side of (4.16) is estimated similarly. Hence we obtain

$$
\left\langle \frac{\partial w}{\partial t} \right\rangle_x^{(\alpha)} \le C_5(\varepsilon_0,T)\langle g\rangle_x^{(\alpha)}. \qquad (4.19)
$$

Next the difference with respect to t of $w^\dagger$ is expressed for $t'\le t$:

$$
\begin{aligned}
w^\dagger(x_1,t)-w^\dagger(x_1,t') &= \int_{2t'-t}^t \mathrm{d}\tau \int_{-\infty}^\infty \frac{\partial}{\partial t} Z_\varepsilon(x_1-y,t-\tau)(g(y,\tau)-g(x_1,\tau))\,\mathrm{d}y \\
&- \int_{2t'-t}^{t'} \mathrm{d}\tau \int_{-\infty}^\infty \frac{\partial}{\partial t'} Z_\varepsilon(x_1-y,t'-\tau)(g(y,\tau)-g(x_1,\tau))\,\mathrm{d}y \\
&+ \int_{-\infty}^{2t'-t} \mathrm{d}\tau \int_{-\infty}^\infty \left(\frac{\partial}{\partial t} Z_\varepsilon(x_1-y,t-\tau) - \frac{\partial}{\partial t'} Z_\varepsilon(x_1-y,t'-\tau) \right) (g(y,\tau)-g(x_1,\tau))\,\mathrm{d}y \\
&\equiv \sum_{j=1}^3 I_j'.
\end{aligned}
$$

Lemma 4.1 yields

$$|I_1'| \le C_6''\langle g\rangle_x^{(\alpha)} \int\limits_{2t'-t}^{t} \mathrm{d}\tau \int\limits_{-\infty|}^{\infty} \frac{|x_1-y|^\alpha(1+\sqrt{\varepsilon(t-\tau)})}{(x_1-y)^2+(t-\tau)^2}\,\mathrm{d}y$$

$$\le C_6'\langle g\rangle_x^{(\alpha)} \int\limits_{2t'-t}^{t} \frac{1+\sqrt{\varepsilon(t-\tau)}}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau \le C_6(\varepsilon_0,T)\langle g\rangle_x^{(\alpha)}|t-t'|^\alpha.$$

I_2' is estimated by just the same way as I_1'. For I_3' we do in the same way with the help of the mean value theorem and the estimate of $\partial^2 Z_\varepsilon(x_1,t)/\partial t^2$:

$$\left|\frac{\partial^2}{\partial t^2}Z_\varepsilon(x_1,t)\right| \le C_7\frac{\varepsilon}{t^2}\exp\left[-\frac{\varepsilon x_1^2}{4t}\right] + C_7'\int\limits_0^{t/\varepsilon} z^{-5/2}\exp\left[-\frac{x_1^2}{4z}-\frac{(t-\varepsilon z)^2}{4z}\right]\mathrm{d}z.$$

After some lengthy calculations we get

$$\langle w^\dagger\rangle_t^{(\alpha)} \le C_8(\varepsilon_0,T)\langle g\rangle_x^{(\alpha)}.$$

Second term in the right hand side of (4.16) is estimated similarly. Hence we obtain

$$\left\langle\frac{\partial w}{\partial t}\right\rangle_t^{(\alpha)} \le C_9(\varepsilon_0,T)\big(\langle g\rangle_x^{(\alpha)}+\langle g\rangle_t^{(\alpha)}\big). \tag{4.20}$$

From (4.16) we can derive

$$\frac{\partial^2}{\partial t\partial x_1}w(x_1,t) = \int\limits_0^t \mathrm{d}\tau \int\limits_{-\infty}^{\infty} \frac{\partial}{\partial t}Z_\varepsilon(x_1-y,t-\tau)\left(\frac{\partial}{\partial y}g(y,\tau)-\frac{\partial}{\partial x_1}g(x_1,\tau)\right)\mathrm{d}y$$

$$+\int\limits_0^t \frac{\partial}{\partial x_1}g(x_1,\tau)\,\mathrm{d}\tau\int\limits_{-\infty}^{\infty}\frac{\partial}{\partial t}Z_\varepsilon(x_1-y,t-\tau)\,\mathrm{d}y + \frac{\partial}{\partial x_1}g(x_1,t).$$

Then, repeating the above arguments, we obtain the estimates

$$\left\langle\frac{\partial^2 w}{\partial t\partial x_1}\right\rangle_x^{(\alpha)} \le C_{10}(\varepsilon_0,T)\left\langle\frac{\partial g}{\partial x_1}\right\rangle_x^{(\alpha)}, \quad \left\langle\frac{\partial^2 w}{\partial t\partial x_1}\right\rangle_t^{(\alpha/2)} \le C_{10}(\varepsilon_0,T)\|g\|_{1+\alpha}. \tag{4.21}$$

We come to the estimate $[\partial w/\partial t]^{(\alpha,\alpha)}$. Since $w^\dagger$ can be written as

$$w^\dagger(x_1,t) = \int\limits_{-\infty}^{\infty}\mathrm{d}\tau\int\limits_{-\infty}^{\infty}\frac{\partial}{\partial t}Z_\varepsilon(x_1-y,\tau)\,(g(y,t-\tau)-g(x_1,t-\tau))\,\mathrm{d}y,$$

$$\frac{w^\dagger(x_1,t)-w^\dagger(x_1,t-\Delta t)}{(\Delta t)^\alpha} \equiv W^\dagger(x_1,t)$$

$$= \int\limits_{-\infty}^{\infty}\mathrm{d}\tau\int\limits_{-\infty}^{\infty}\frac{\partial}{\partial t}Z_\varepsilon(x_1-y,\tau)\,(\varphi(y,t-\tau)-\varphi(x_1,t-\tau))\,\mathrm{d}y$$

$$= \int_0^t \mathrm{d}\tau \int_{-\infty}^{\infty} \frac{\partial}{\partial t} Z_\varepsilon(x_1 - y, t-\tau)\left(\varphi(y,\tau) - \varphi(x_1,\tau)\right) \mathrm{d}y$$

for $\Delta t > 0$, where

$$\varphi(x_1, t) = \frac{g(x_1,t) - g(x_1, t-\Delta t)}{(\Delta t)^\alpha}.$$

Again the above argument implies

$$\langle W^\dagger\rangle_x^{(\alpha)} \le C_{11}(\varepsilon_0, T)\langle\varphi\rangle_x^{(\alpha)} = C_{11}(\varepsilon_0,T)\sup_{x_1,x_1',t}\frac{|\varphi(x_1,t)-\varphi(x_1',t)|}{|x_1-x_1'|^\alpha}$$
$$= C_{11}(\varepsilon_0,T)\sup_{x_1,x_1',t}\frac{|g(x_1,t)-g(x_1',t)-g(x_1,t-\Delta t)+g(x_1',t-\Delta t)|}{|x_1-x_1'|^\alpha(\Delta t)^\alpha},$$

and hence

$$[w^\dagger]^{(\alpha,\alpha)} \le C_{11}(\varepsilon_0,T)[g]^{(\alpha,\alpha)}.$$

Second term in the right hand side of (4.16) is estimated similarly. Thus we obtain

$$\left[\frac{\partial w}{\partial t}\right]^{(\alpha,\alpha)} \le C_{11}(\varepsilon_0,T)[g]^{(\alpha,\alpha)}. \tag{4.22}$$

Similarly we have

$$\left[\frac{\partial^2 w}{\partial t\partial x_1}\right]^{(\alpha,\alpha/2)} \le C_{12}(\varepsilon_0,T)\left[\frac{\partial g}{\partial x_1}\right]^{(\alpha,\alpha/2)}. \tag{4.23}$$

Now it is necessary to get the representation of $(\mathscr{F}\mathscr{L})^{-1}[r_\varepsilon\tilde g_2]$:

$$(\mathscr{F}\mathscr{L})^{-1}[r_\varepsilon\tilde g_2] = 2\frac{\partial^2}{\partial x_2^2}\int_0^t \mathrm{d}\tau\int_{-\infty}^{\infty}\Gamma_\varepsilon(x_1-y,x_2,t-\tau)g_2(y,\tau)\,\mathrm{d}y\bigg|_{x_2=0}$$
$$= 2\frac{\partial^2}{\partial x_2^2}\int_0^{t'} \mathrm{d}\tau\int_{-\infty}^{\infty}\Gamma_1(x_1-y,x_2,t'-\tau)G_2(y,\tau)\,\mathrm{d}y\bigg|_{x_2=0},$$

where $t' = t/\varepsilon,\ G_2(x_1,t') = g_2(x_1,t),\ \Gamma_1 = \Gamma_\varepsilon|_{\varepsilon=1}$ (see [25]). From this we have

$$\langle(\mathscr{F}\mathscr{L})^{-1}[r_\varepsilon\tilde g_2]\rangle^{(\alpha,\alpha/2)} \le C_{13}\langle G_2\rangle^{(1+\alpha,(1+\alpha)/2)}$$
$$= C_{13}(\langle g_2\rangle_x^{(1+\alpha)} + \varepsilon^{(1+\alpha)/2}\langle g_2\rangle_t^{((1+\alpha)/2)}).$$

Equation (4.9) and estimates (4.19)–(4.23) lead to

$$\left\|\frac{\partial\varrho}{\partial t}\right\|_{1+\alpha} \le C_{14}(\varepsilon_0,T)\left(\|g_1\|_{1+\alpha} + \|g_2\|_{2+\alpha}\right). \tag{4.24}$$

The same arguments can be applied to (4.17) with the help of Lemma 4.1, so that similar estimates to (4.19), (4.20) and (4.22) hold for $\partial^2 w/\partial x_1^2$:

$$\left\|\frac{\partial^2 w}{\partial x_1^2}\right\|_\alpha \le C_{15}(\varepsilon_0,T)\left\|\frac{\partial g}{\partial x_1}\right\|_\alpha;$$

moreover, we have

$$\left\langle \frac{\partial^2 w}{\partial x_1^2} \right\rangle_t^{(\alpha)} \le C_{16}(\varepsilon_0, T) \left\langle \frac{\partial g}{\partial x_1} \right\rangle_x^{(\alpha)}.$$

Lower order derivatives of w are easily estimated. Therefore, from (4.9) we deduce

$$\|\varrho\|_{2+\alpha} \le C_{17}(\varepsilon_0, T) \left(\|g_1\|_{1+\alpha} + \|g_2\|_{2+\alpha} \right),$$
$$\left\langle \frac{\partial^2 \varrho}{\partial x_1^2} \right\rangle_t^{(\alpha)} \le C_{18}(\varepsilon_0, T) \Bigg(\left\langle \frac{\partial g_1}{\partial x_1} \right\rangle_x^{(\alpha)} + \left\langle \frac{\partial g_2}{\partial x_1} \right\rangle^{(1+\alpha,(1+\alpha)/2)} \Bigg). \tag{4.25}$$

Estimates (4.24) and (4.25) easily give the estimate for $v(x_1, t)$ in (4.10):

$$\left\| \frac{\partial v}{\partial t} \right\|_\alpha + \|v\|_{2+\alpha} \le C_{19} \left(\|g_1\|_{1+\alpha} + \|g_2\|_{2+\alpha} + \left\| \frac{\partial g_2}{\partial t} \right\|_\alpha \right). \tag{4.26}$$

For $u(x,t)$ we prepare two representations:

$$u(x,t) = -2 \int_0^t \mathrm{d}\tau \int_{-\infty}^{\infty} \frac{\partial}{\partial x_2} \Gamma_\varepsilon(x_1 - y, x_2, t - \tau) v(y, \tau)\, \mathrm{d}y, \tag{4.27}$$

$$u(x,t) = -2 \int_0^t \mathrm{d}\tau \int_{-\infty}^{\infty} \Gamma_\varepsilon(x_1 - y, x_2, t - \tau) g(y, \tau)\, \mathrm{d}y, \quad g = \frac{1}{b} \left(-g_1 + \frac{\partial \varrho}{\partial t} \right) \tag{4.28}$$

since $\tilde{v}$ is represented as

$$\tilde{v} = \tilde{g}_2 + d\tilde{\varrho} = \frac{1}{br_\varepsilon}(-\tilde{g}_1 + s\tilde{\varrho}).$$

Here we use (4.27) for the estimates of $\partial^2 u/\partial x_1^2$, $\partial^2 u/\partial x_2^2$, $\partial u/\partial x_1$, and (4.28) for the estimates of $\partial^2 u/\partial x_1 \partial x_2$, $\partial u/\partial x_2$.

Introduce the notation with $t' = t/\varepsilon$

$$u(x, \varepsilon t') = U(x, t'), \quad v(x_1, \varepsilon t') = V(x_1, t'), \quad g(x_1, \varepsilon t') = G(x_1, t').$$

Then, (4.27) and (4.28) are represented as

$$U(x, t') = -2 \frac{\partial \Gamma_1}{\partial x_2} * V \quad \text{and} \quad U(x, t') = -2\Gamma_1 * G,$$

respectively. We can find the following estimates (see [26]):

$$\left\langle \frac{\partial \Gamma_1}{\partial x_2} * V \right\rangle_{t'}^{(\alpha/2)} \le C_{20} \langle V \rangle_{t'}^{(\alpha/2)}, \quad \left\langle \frac{\partial \Gamma_1}{\partial x_2} * V \right\rangle_x^{(\alpha)} \le C_{20} \big(\langle V \rangle_x^{(\alpha)} + \langle V \rangle_{t'}^{(\alpha/2)} \big).$$

Simultaneously we have

$$\left\langle \frac{\partial^2 U}{\partial x_1^2} \right\rangle_{t'}^{(\alpha/2)} + \left\langle \frac{\partial^2 U}{\partial x_2^2} \right\rangle_{t'}^{(\alpha/2)} \le C_{21} \Bigg(\left\langle \frac{\partial^2 V}{\partial x_1^2} \right\rangle_{t'}^{(\alpha/2)} + \left\langle \frac{\partial V}{\partial t'} \right\rangle_{t'}^{(\alpha/2)} \Bigg),$$

$$\left\langle \frac{\partial^2 U}{\partial x_1^2} \right\rangle_x^{(\alpha)} + \left\langle \frac{\partial^2 U}{\partial x_2^2} \right\rangle_x^{(\alpha)} \le C_{21} \left(\left\langle \frac{\partial^2 V}{\partial x_1^2} \right\rangle_x^{(\alpha)} + \left\langle \frac{\partial^2 V}{\partial x_1^2} \right\rangle_{t'}^{(\alpha/2)} + \left\langle \frac{\partial V}{\partial t'} \right\rangle_x^{(\alpha)} + \left\langle \frac{\partial V}{\partial t'} \right\rangle_{t'}^{(\alpha/2)} \right).$$

These inequalities yield

$$\left\langle \frac{\partial^2 u}{\partial x_1^2} \right\rangle_t^{(\alpha/2)} + \left\langle \frac{\partial^2 u}{\partial x_2^2} \right\rangle_t^{(\alpha/2)} \le C_{21}\left(\left\langle \frac{\partial^2 v}{\partial x_1^2} \right\rangle_t^{(\alpha/2)} + \varepsilon \left\langle \frac{\partial v}{\partial t} \right\rangle_t^{(\alpha/2)} \right),$$

$$\begin{aligned} \left\langle \frac{\partial^2 u}{\partial x_1^2} \right\rangle_x^{(\alpha)} + \left\langle \frac{\partial^2 u}{\partial x_2^2} \right\rangle_x^{(\alpha)} &\le C_{21}\left(\left\langle \frac{\partial^2 v}{\partial x_1^2} \right\rangle_x^{(\alpha)} + \varepsilon^{\alpha/2} \left\langle \frac{\partial^2 v}{\partial x_1^2} \right\rangle_t^{(\alpha/2)} \right. \\ &\quad \left. + \varepsilon \left\langle \frac{\partial v}{\partial t} \right\rangle_x^{(\alpha)} + \varepsilon^{1+\alpha/2} \left\langle \frac{\partial v}{\partial t} \right\rangle_t^{(\alpha/2)} \right). \end{aligned} \tag{4.29}$$

Furthermore, by the same method as that for $w^\dagger$ we have

$$\begin{aligned} \left[\frac{\partial^2 U}{\partial x_1^2} \right]^{(\alpha,\alpha/2)} &\le C'_{22}\left(\left[\frac{\partial^2 V}{\partial x_1^2} \right]^{(\alpha,\alpha/2)} + \sup_{x_1,t',s,\Delta t'} \frac{1}{|\Delta t'|^{\alpha/2}|s|^{\alpha/2}} \left| \frac{\partial^2 V}{\partial x_1^2}(x_1, t'-s) \right.\right. \\ &\quad \left.\left. - \frac{\partial^2 V}{\partial x_1^2}(x_1,t') - \frac{\partial^2 V}{\partial x_1^2}(x_1, t'+\Delta t' - s) + \frac{\partial^2 V}{\partial x_1^2}(x_1, t'+\Delta t') \right| \right) \\ &\le C_{22}\left(\left[\frac{\partial^2 V}{\partial x_1^2} \right]^{(\alpha,\alpha/2)} + \left\langle \frac{\partial^2 V}{\partial x_1^2} \right\rangle_{t'}^{(\alpha)} \right), \end{aligned}$$

and hence

$$\left[\frac{\partial^2 u}{\partial x_1^2} \right]^{(\alpha,\alpha/2)} \le C_{23}\left(\left[\frac{\partial^2 v}{\partial x_1^2} \right]^{(\alpha,\alpha/2)} + \varepsilon^{\alpha/2} \left\langle \frac{\partial^2 v}{\partial x_1^2} \right\rangle_t^{(\alpha)} \right). \tag{4.30}$$

Analogously we obtain the estimate for $[\partial^2 u/\partial x_2^2]^{(\alpha,\alpha/2)}$.

By virtue of the expression (4.28) we get

$$\frac{\partial U}{\partial x_2} = -2\frac{\partial \Gamma_1}{\partial x_2} * G, \quad \frac{\partial^2 U}{\partial x_2 \partial x_1} = -2\frac{\partial \Gamma_1}{\partial x_2} * \frac{\partial G}{\partial x_1},$$

from which it follows

$$\left\langle \frac{\partial U}{\partial x_2} \right\rangle_{t'}^{(\alpha)} \le C_{24}\langle G \rangle_{t'}^{(\alpha)}, \quad \left\langle \frac{\partial^2 U}{\partial x_1 \partial x_2} \right\rangle_{t'}^{(\alpha/2)} \le C_{24}\left\langle \frac{\partial G}{\partial x_1} \right\rangle_{t'}^{(\alpha/2)},$$

and hence

$$\left\langle \frac{\partial u}{\partial x_2} \right\rangle_t^{(\alpha)} \le C_{25}\langle g \rangle_t^{(\alpha)}, \quad \left\langle \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\rangle_t^{(\alpha/2)} \le C_{25}\left\langle \frac{\partial g}{\partial x_1} \right\rangle_t^{(\alpha/2)}. \tag{4.31}$$

Each term in the right hand side of (4.29)–(4.31) is estimated by virtue of (4.24)–(4.26).

Similar arguments are applicable to other terms appearing in the norm $\|u\|_{2+\alpha}$. And finally from the first equation in (4.4) we derive the uniform estimate $\varepsilon\|\partial u/\partial t\|_\alpha$ with respect to ε.

Lemma 4.2. *Let ε_0 be a fixed positive number, and b and d are positive constants. Assume that $g_1 \in \underset{0}{E}^{1+\alpha}(\mathbb{R}_T)$ and $g_2 \in \underset{0}{\check{E}}^{2+\alpha}(\mathbb{R}_T)$ with $\alpha \in (0,1)$ and $T > 0$. Then problem (4.4) with any $\varepsilon \in (0, \varepsilon_0]$ has a unique solution*

$$u \in \underset{0}{P_\varepsilon}^{2+\alpha}(\mathbb{R}^2_{+,T}), \quad \varrho \in \underset{0}{\widehat{E}}^{2+\alpha}(\mathbb{R}_T) \quad \big(\mathbb{R}^2_{+,T} \equiv \mathbb{R}^2_+ \times [0,T],\ \mathbb{R}_T \equiv \mathbb{R} \times [0,T]\big)$$

satisfying the inequality

$$\varepsilon\left\|\frac{\partial u}{\partial t}\right\|_{\alpha}+\|u\|_{2+\alpha}+\|\varrho\|_{2+\alpha}+\left\|\frac{\partial \varrho}{\partial t}\right\|_{1+\alpha} \leq C_{26}\left(\|g_1\|_{1+\alpha}+\|g_2\|_{2+\alpha}\left\|\frac{\partial g_2}{\partial t}\right\|_{\alpha}\right), \quad (4.32)$$

where C_{26} is a positive constant depending on ε_0, but not on ε.

Now we shall solve problem (4.1) by the regularizer method.

In the following (θ, r) corresponds to the above (x_1, x_2). For a suitably small positive number λ we introduce two coverings $\{\omega^k\}$ and $\{\Omega^k\}$ of $\overline{\Omega}$ as follows (cf. [27, 28]): Let k be in $\mathscr{M}_1$ if two dimensional squares ω^k and Ω^k included completely in Ω are mapped by Π_1 from those with common center (r_k, θ_k) and with the length of their edges, in parallel directions of axes, equal to $\lambda/2$ and λ, respectively. For $k \in \mathscr{M}_2$ satisfying $\omega^k \cap \Gamma \neq \varnothing$, ω^k and $\Omega^k \subset \overline{\Omega}$ are defined in the local coordinates (z_1, z_2) with an origin at $(r_k, \theta_k) \in \Gamma$ by

$$\omega^k = \Pi_2\left\{|z_2| \leq \lambda/2,\ 0 \leq z_1 - F_\Gamma(z_2) \leq \lambda\right\},$$

$$\Omega^k = \Pi_2\left\{|z_2| \leq \lambda,\ 0 \leq z_1 - F_\Gamma(z_2) \leq 2\lambda\right\},$$

where equation $z_1 = F_\Gamma(z_2)$ represents Γ around $(r_k, \theta_k) \in \Gamma$ and Π_2 is a transformation from (z_1, z_2) to $(r, \theta) \in \Omega^k$. By $\{\omega^k\}$ and $\{\Omega^k\}$, $k \in \mathscr{M}_3$, we denote the coverings of Γ_* in $\overline{\Omega}$ and by Π_3 the transformation from z to $(r, \theta) \in \Omega^k$.

Let the partitions of unity $\{\eta_k\}$ and $\{\eta_k^*\}$ subordinated to $\{\omega^k\}$ and $\{\Omega^k\}$ be such as

$$\eta_k(r,\theta) = \begin{cases} 1 & \text{for } (r,\theta) \in \omega^k, \\ 0 & \text{for } (r,\theta) \in \bar{\Omega} \setminus \Omega^k, \end{cases} \qquad 0 \leq \eta_k(r,\theta) \leq 1, \quad \eta_k \in C_0^\infty(\overline{\Omega}),$$

$$\left|\mathrm{D}_{r,\theta}^j \eta_k(r,\theta)\right| \leq C_{27}\lambda^{-|j|}, \quad \eta_k^*(r,\theta) \equiv \frac{\eta_k(r,\theta)}{\sum_k (\eta_k(r,\theta))^2}.$$

Obviously, $\{\eta_k^*(r,\theta)\}$ have properties

$$\eta_k^*(r,\theta) = 0 \text{ if } (r,\theta) \in \overline{\Omega} \setminus \Omega^k, \quad \sum_k \eta_k(r,\theta)\eta_k^*(r,\theta) = 1, \quad \left|\mathrm{D}_{r,\theta}^j \eta_k^*(r,\theta)\right| \leq C_{28}\lambda^{-|j|}.$$

First let $T_* = \gamma\lambda^2$, where $\gamma \in (0,1)$ will be specified later. Then the regularizer $\mathscr{R}$ is defined by

$$\mathscr{R}H = \sum_{m=1,2,3}\sum_{k\in\mathscr{M}_m} \eta_k^* u^k + \sum_{k\in\mathscr{M}_2} \eta_k^* \varrho^k, \quad H = (\phi, \psi_*, \psi_1, \psi_2).$$

Here $u^k = \Pi_1 \bar{u}^k$, where $\bar{u}^k$ is a solution of problem

$$\begin{aligned} \frac{\partial \bar{u}^k}{\partial t} - \Pi_1^{-1}\left(\mathscr{L}_{*,k}\right)\bar{u}^k &= \Pi_1^{-1}\eta_k\phi \quad \text{in } \mathbb{R}^2 \times (0,T), \\ \bar{u}^k|_{t=0} &= 0 \quad \text{on } \mathbb{R}^2 \end{aligned} \qquad (4.33)$$

$$(\mathscr{L}_{*,k} \equiv \mathscr{L}_*(r_k, \theta_k; \partial/\partial r, \partial/\partial\theta)|_{t=0});$$

$u^k = \Pi_3 \bar{u}^k$, where $\bar{u}^k$ is a solution of problem

$$
\begin{aligned}
&\frac{\partial \bar{u}^k}{\partial t} - \Pi_3^{-1} \left(\mathscr{L}_{*,k}\right) \bar{u}^k = \Pi_3^{-1} \eta_k \phi \quad \text{in } \ \mathbb{R}^2_+ \times (0,T), \\
&\bar{u}^k|_{t=0} = 0 \quad \text{on } \ \mathbb{R}^2_+ \equiv \{(z_1, z_2) \in \mathbb{R}^2 \mid z_1 > 0\}, \\
&\left.\frac{\partial \bar{u}^k}{\partial z_1}\right|_{z_1=0} = \Pi_3^{-1} \eta_k \psi_*;
\end{aligned}
\tag{4.34}
$$

$u^k = \Pi_2 \bar{u}^k$, $\varrho^k = \Pi_2 \bar{\varrho}^k$, where $(\bar{u}^k, \bar{\varrho}^k)$ is a solution of problem

$$
\begin{aligned}
&\frac{\partial \bar{u}^k}{\partial t} - \Pi_2^{-1} \left(\mathscr{L}_{*,k}\right) \bar{u}^k = \Pi_2^{-1} \eta_k \phi \quad \text{in } \ \mathbb{R}^2_+ \times (0,T), \\
&(\bar{u}^k, \bar{\varrho}^k)|_{t=0} = 0, \\
&\frac{\partial \bar{\varrho}^k}{\partial t} - b_k \left.\frac{\partial \bar{u}^k}{\partial z_1}\right|_{z_1=0} = \Pi_2^{-1} \eta_k \psi_1, \quad \bar{u}^k - d_k \bar{\varrho}^k|_{z_1=0} = \Pi_2^{-1} \eta_k \psi_2
\end{aligned}
\tag{4.35}
$$

with

$$
b_k \mathbf{e}_{z_1} = \Pi_2 (b_2, b_1)(\bar{\zeta})|_{\theta=\theta_k, t=0}, \quad d_k = d(\bar{\zeta}))|_{\theta=\theta_k, t=0}, \quad \mathbf{e}_{z_1} = (1,0).
$$

Then it is not difficult to see that $\mathscr{R}H = (u', \varrho')$ satisfies

$$
\begin{aligned}
&\varepsilon \frac{\partial u'}{\partial t} - \mathscr{L}_* u' = \phi - \mathscr{T}_1 H \quad \text{in } \ \Omega, \ t > 0, \\
&\frac{\partial u'}{\partial r} = \psi_* - \mathscr{T}_2 H \quad \text{on } \ \Gamma_*, \ t > 0, \\
&\frac{\partial \varrho'}{\partial t} - b_2(\bar{\zeta}) \frac{\partial u'}{\partial r} - b_1(\bar{\zeta}) \frac{\partial u'}{\partial \theta} = \psi_1 - \mathscr{T}_3 H, \\
&u' - d(\bar{\zeta}) \varrho' = \psi_2 - \mathscr{T}_4 H \quad \text{on } \ \Gamma, \ t > 0, \\
&(\bar{u}', \bar{\varrho}')|_{t=0} = 0.
\end{aligned}
\tag{4.36}
$$

Here the operator $\mathscr{T} = (\mathscr{T}_1, \mathscr{T}_2, \mathscr{T}_3, \mathscr{T}_4)$ is defined on

$$
\mathscr{H}_T = \underset{0}{E}^{\alpha}(\overline{Q}_T) \times \underset{0}{E}^{1+\alpha}(\Gamma_{*T}) \times \underset{0}{E}^{1+\alpha}(\Gamma_T) \times \underset{0}{\check{E}}^{2+\alpha}(\Gamma_T)
$$

by the formulae

$$
\begin{aligned}
\mathscr{T}_1 H &= \mathscr{L}_* u' - \sum_{j=1,2,3} \sum_{k \in \mathscr{M}_j} \eta_k^* \Pi_j \mathscr{L}_{*,k} \bar{u}^k, \\
\mathscr{T}_2 H &= \frac{\partial u'}{\partial r} - \sum_{k \in \mathscr{M}_3} \eta_k^* \Pi_3 \frac{\partial \bar{u}^k}{\partial z_1}, \\
\mathscr{T}_3 H &= b_2(\bar{\zeta}) \frac{\partial u'}{\partial r} + b_1(\bar{\zeta}) \frac{\partial u'}{\partial \theta} - \sum_{k \in \mathscr{M}_2} \eta_k^* \Pi_2 b_k \frac{\partial \bar{u}^k}{\partial z_1} \\
\mathscr{T}_4 H &= -d(\bar{\zeta}) \varrho' + \sum_{k \in \mathscr{M}_2} \eta_k^* \Pi_2 d_k \bar{\varrho}^k.
\end{aligned}
$$

One can find the solution of problem (4.1) in the form

$$
(u, \varrho) = \mathscr{R}(\mathscr{I} + \mathscr{T} + \mathscr{T}^2 + \dots) H = \mathscr{R}(\mathscr{I} - \mathscr{T})^{-1} H \quad (\mathscr{I} \text{ is an identity operator}),
$$

for which it is necessary to show that the operator $\mathscr{T}$ is a contraction on $\mathscr{H}_T$. We first note that $\Gamma \in C^{2+\alpha}$ implies that $F_\Gamma \in C^{2+\alpha}(\bar{B}_\delta)$ ($B_\delta \equiv \{z_2 \in \mathbb{R} \mid |z_2| < \delta\}$) and $F_\Gamma(0) = 0$, $\partial F_\Gamma/\partial z_2(0) = 0$, $\|F_\Gamma\|_{C^{2+\alpha}(\bar{B}_\delta)} \le C$ with some constants δ and C being independent of z_2. We take λ small enough to satisfy $\lambda \le \delta/2$. Clearly,

$$|F_\Gamma(z_2)| \le C|z_2|^{1+\alpha}, \quad \left|\frac{\partial F_\Gamma(z_2)}{\partial z_2}\right| \le C|z_2|^\alpha. \tag{4.37}$$

We estimate each term in $\mathscr{T}_1 H$, $\mathscr{T}_2 H$, $\mathscr{T}_3 H$, $\mathscr{T}_4 H$ in such a way that for the lower order terms the interpolation inequalities, for example,

$$\sup_{x\in\Omega} |\nabla u(x)| \le c\big(\langle u\rangle_x^{(\alpha)}\big)^{(1+\alpha)/2}\big(\langle u\rangle_x^{(2+\alpha)}\big)^{(1-\alpha)/2}, \quad \langle \nabla^2 u\rangle_t^{(\alpha/2)} \le c\lambda^\alpha [\nabla^2 u]^{(\alpha,\alpha/2)}$$

for $u \in \underset{0}{E}^{2+\alpha}(\overline{Q}_T)$ are used, while for the highest order terms the smallness of their coefficients like (4.37) derived from the smallness of λ and γ are used. Then we get the following estimate after some lengthy, but straightforward calculations:

$$\begin{aligned}\|\mathscr{T} H\|_{\mathscr{H}_t} &\equiv \|\mathscr{T}_1 H\|_{E^\alpha(\overline{Q}_t)} + \|\mathscr{T}_2 H\|_{E^{1+\alpha}(\Gamma_{*t})} + \|\mathscr{T}_3 H\|_{E^{1+\alpha}(\Gamma_t)} + \|\mathscr{T}_4 H\|_{\check{E}^{2+\alpha}(\Gamma_t)} \\ &\le C_{29}(\lambda,\gamma)\Big(\sum_{j=1,2,3}\sum_{k\in\mathscr{M}_j} \|u^k\|_{E^{2+\alpha}(\overline{Q}_t^k)} + \sum_{k\in\mathscr{M}_2} \|\varrho^k\|_{\widehat{E}^{2+\alpha}(\Gamma_t^k)}\Big)\end{aligned} \tag{4.38}$$

for any $t \in (0, T_*)$, where $Q_t^k = \Omega^k \times (0,t), \Gamma_t^k = \Gamma_t \cap Q_t^k$ and

$$C_{29}(\lambda,\gamma) = C_{29,1}(\lambda)C_{29,2}(\gamma) + C_{29,3}(\lambda) + C_{29,4}(\gamma)$$

is a positive constant with $C_{29,j}(\cdot)$ being dependent non-decreasingly on their argument and $C_{29,j}(0) = 0$ ($j = 1,2,3,4$). By virtue of Lemma 4.2 and (4.7) we see that $\mathscr{T}$ is the contraction operator on $\mathscr{H}_{T_*}$ for suitably small λ and γ. Hence problem (4.1) is solvable on $[0, \gamma\lambda^2]$. In the same way one can solve problem (4.1) on $[\gamma\lambda^2, 2\gamma\lambda^2]$. Repeating the above argument finite times, we finally obtain the solution on $[0,T]$ with arbitrary $T > 0$. Summing up the above leads to

Lemma 4.3. *For any $T > 0$ and any $\varepsilon_0 > 0$ problem* (4.1) *with a fixed $\varepsilon \in (0, \varepsilon_0]$ has a unique solution $(u, \varrho) \in \underset{0}{P_\varepsilon}^{2+\alpha}(\overline{Q}_T) \times \underset{0}{\widehat{E}}^{2+\alpha}(\Gamma_T)$ satisfying*

$$\|u\|_{P_\varepsilon^{2+\alpha}(\overline{Q}_T)} + \|\varrho\|_{\widehat{E}^{2+\alpha}(\Gamma_T)} \le C_{30}\|H\|_{\mathscr{H}_T}, \tag{4.39}$$

where C_{30} is a positive constant depending on ε_0, but not on ε.

5. Nonlinear Problem: Proof of Theorem 2.2

In this section we construct the solution to problem (3.2) by the successive approximation method.

Let (p_n^*, ζ_n^*) ($n = 1,2,3,\dots$) be a solution of problem (4.1) with

$$\phi = \Phi^\varepsilon(p_{n-1}^*, \zeta_{n-1}^*), \quad \psi_* = \Psi_*(\zeta_{n-1}^*), \quad \psi_1 = \Psi_1(p_{n-1}^*, \zeta_{n-1}^*), \quad \psi_2 = \Psi_2$$

for a given $(p_{n-1}^*, \zeta_{n-1}^*) \in \underset{0}{P_\varepsilon}^{2+\alpha}(\overline{Q}_T) \times \underset{0}{\widehat{E}}^{2+\alpha}(\Gamma_T)$, and $(p_0^*, \zeta_0^*) = (0,0)$.

It is easily seen that there exists a constant $M > 0$ dependent on ε_0 and T, but not on ε such that

$$\|(\Phi^\varepsilon(0,0), \Psi_*(0), \Psi_1(0,0), \Psi_2)\|_{\mathscr{H}_T} \le M. \tag{5.1}$$

From the same argument as in (4.38) we derive the following inequality with the help of (5.1):

$$\begin{aligned} \|(\Phi^\varepsilon(p^*_{n-1}, \zeta^*_{n-1}), \Psi_*(\zeta^*_{n-1}), \Psi_1(p^*_{n-1}, \zeta^*_{n-1}), \Psi_2)\|_{\mathscr{H}_t} \\ \le \|(\Phi^\varepsilon(0,0), \Psi_*(0), \Psi_1(0,0), \Psi_2)\|_{\mathscr{H}_t} \\ + \|(\Phi^\varepsilon(p^*_{n-1}, \zeta^*_{n-1}) - \Phi^\varepsilon(0,0), \Psi_*(\zeta^*_{n-1}) - \Psi_*(0), \Psi_1(p^*_{n-1}, \zeta^*_{n-1}) - \Psi_1(0,0), 0)\|_{\mathscr{H}_t} \\ \le M + \beta(\|p^*_{n-1}\|_{E^{2+\alpha}(\overline{Q}_t)} + \|\zeta^*_{n-1}\|_{\widehat{E}^{2+\alpha}(\Gamma_t)}) \\ + C_\beta t^\chi F(\|p^*_{n-1}\|_{E^{2+\alpha}(\overline{Q}_t)} + \|\zeta^*_{n-1}\|_{\widehat{E}^{2+\alpha}(\Gamma_t)}) \end{aligned} \tag{5.2}$$

for any $t \in (0,T)$ and any $\beta > 0$, where C_β is a positive constant depending on β non-increasingly, $\chi > 0$ is a constant depending on α (possibly) and $F(\cdot)$ is a polynomial in its argument. Thus, Lemma 4.3 yields

$$\begin{aligned} \|p^*_n\|_{P^{2+\alpha}_\varepsilon(\overline{Q}_t)} + \|\zeta^*_n\|_{\widehat{E}^{2+\alpha}(\Gamma_t)} \le C_{30}(M + \beta(\|p^*_{n-1}\|_{E^{2+\alpha}(\overline{Q}_t)} + \|\zeta^*_{n-1}\|_{\widehat{E}^{2+\alpha}(\Gamma_t)}) \\ + C_\beta t^\chi F(\|p^*_{n-1}\|_{E^{2+\alpha}(\overline{Q}_t)} + \|\zeta^*_{n-1}\|_{\widehat{E}^{2+\alpha}(\Gamma_t)})) \end{aligned} \tag{5.3}$$

for any $t \in (0,T)$. Now we take first $\beta = 1/(4C_{30})$, and then

$$T' = \min\left\{T, \left(\frac{M}{2C_\beta F(2C_{30}M)}\right)^{1/\chi}\right\}.$$

Therefore, from (5.3) it follows that

$$\|p^*_n\|_{P^{2+\alpha}_\varepsilon(\overline{Q}_t)} + \|\zeta^*_n\|_{\widehat{E}^{2+\alpha}(\Gamma_t)} \le 2C_{30}M$$

for any $t \in (0,T')$ if

$$\|p^*_{n-1}\|_{P^{2+\alpha}_\varepsilon(\overline{Q}_T)} + \|\zeta^*_{n-1}\|_{\widehat{E}^{2+\alpha}(\Gamma_T)} \le 2C_{30}M,$$

that is, for $n = 0, 1, 2, \dots$

$$\begin{gathered} (p^*_n, \zeta^*_n) \in \underset{0}{P_\varepsilon}^{2+\alpha}(\overline{Q}_{T'}) \times \underset{0}{\widehat{E}}^{2+\alpha}(\Gamma_{T'}), \\ \|p^*_n\|_{P^{2+\alpha}_\varepsilon(\overline{Q}_{T'})} + \|\zeta^*_n\|_{\widehat{E}^{2+\alpha}(\Gamma_{T'})} \le 2C_{30}M. \end{gathered} \tag{5.4}$$

For the convergence of the sequence $\{(p^*_n, \zeta^*_n)\}$ we consider the difference $(p^*_n - p^*_{n-1}, \zeta^*_n - \zeta^*_{n-1})$, which satisfies (4.1) with

$$u = p^*_n - p^*_{n-1}, \quad \varrho = \zeta^*_n - \zeta^*_{n-1}, \quad \phi = \Phi^\varepsilon(p^*_{n-1}, \zeta^*_{n-1}) - \Phi^\varepsilon(p^*_{n-2}, \zeta^*_{n-2}),$$

$$\psi_* = \Psi_*(\zeta^*_{n-1}) - \Psi_*(\zeta^*_{n-2}), \quad \psi_1 = \Psi_1(p^*_{n-1}, \zeta^*_{n-1}) - \Psi_1(p^*_{n-2}, \zeta^*_{n-2}), \quad \psi_2 = 0.$$

Similarly as above, with the help of the interpolation inequalities we can obtain a similar estimate as (5.2):

$$\|(\Phi^\varepsilon(p^*_{n-1}, \zeta^*_{n-1}) - \Phi^\varepsilon(p^*_{n-2}, \zeta^*_{n-2}), \Psi_*(\zeta^*_{n-1}) - \Psi_*(\zeta^*_{n-2}),$$

$$
\begin{aligned}
&\Psi_1(p^*_{n-1},\zeta^*_{n-1})-\Psi_1(p^*_{n-2},\zeta^*_{n-2}),0)\|_{\mathscr{H}_t}\\
&\le(\beta'+C_{\beta'}t^{\chi'}F'(\|p^*_{n-1}\|_{E^{2+\alpha}(\overline{Q}_t)}+\|p^*_{n-2}\|_{E^{2+\alpha}(\overline{Q}_t)}\\
&+\|\zeta^*_{n-1}\|_{\widehat{E}^{2+\alpha}(\Gamma_t)}+\|\zeta^*_{n-2}\|_{\widehat{E}^{2+\alpha}(\Gamma_t)}))(\|p^*_{n-1}-p^*_{n-2}\|_{E^{2+\alpha}(\overline{Q}_t)}\\
&+\|\zeta^*_{n-1}-\zeta^*_{n-2}\|_{\widehat{E}^{2+\alpha}(\Gamma_t)})
\end{aligned}
$$

for any $t\in(0,T')$ and any $\beta'>0$, where $C_{\beta'}$ is a positive constant depending on β' non-increasingly, $\chi'>0$ is a constant depending on α (possibly) and $F'(\cdot)$ is a polynomial in its argument. Therefore, Lemma 4.3 and (5.4) yield

$$
\begin{aligned}
&\|p^*_n-p^*_{n-1}\|_{P^{2+\alpha}_\varepsilon(\overline{Q}_t)}+\|\zeta^*_n-\zeta^*_{n-1}\|_{\widehat{E}^{2+\alpha}(\Gamma_t)}\\
&\le C_{30}(\beta'+C_{\beta'}t^{\chi'}F'(4C_{30}M))(\|p^*_{n-1}-p^*_{n-2}\|_{E^{2+\alpha}(\overline{Q}_t)}+\|\zeta^*_{n-1}-\zeta^*_{n-2}\|_{\widehat{E}^{2+\alpha}(\Gamma_t)})
\end{aligned}
$$

for any $t\in(0,T')$. Again taking first $\beta'=1/(4C_{30})$, and then

$$
T_0=\min\Big\{T',\Big(\frac{1}{4C_{30}C_{\beta'}F'(4C_{30}M)}\Big)^{1/\chi'}\Big\},
$$

we obtain

$$
\begin{aligned}
&\|p^*_n-p^*_{n-1}\|_{P^{2+\alpha}_\varepsilon(\overline{Q}_t)}+\|\zeta^*_n-\zeta^*_{n-1}\|_{\widehat{E}^{2+\alpha}(\Gamma_t)}\\
&\quad\le\frac12(\|p^*_{n-1}-p^*_{n-2}\|_{E^{2+\alpha}(\overline{Q}_t)}+\|\zeta^*_{n-1}-\zeta^*_{n-2}\|_{\widehat{E}^{2+\alpha}(\Gamma_t)})\quad\text{for } t\in(0,T_0).
\end{aligned}\tag{5.5}
$$

Consequently, the sequence $\{(p^*_n,\zeta^*_n)\}$ converges to (p^*,ζ^*) uniformly in $\underset{0}{P_\varepsilon}^{2+\alpha}(\overline{Q}_{T_0})\times\widehat{E}_0^{\,2+\alpha}(\Gamma_{T_0})$, which satisfies

$$
\|p^*\|_{P^{2+\alpha}_\varepsilon(\overline{Q}_{T_0})}+\|\zeta^*\|_{\widehat{E}^{2+\alpha}(\Gamma_{T_0})}\le 2C_{30}M.\tag{5.6}
$$

The uniqueness of such a solution follows from the similar inequality as (5.5) for two solutions to problem (3.2).

6. Passing to the limit $\varepsilon\to 0$: Proof of Theorem 2.1

Theorem 2.2 means that for any $\varepsilon\in(0,\varepsilon_0]$ there exists a smooth solution $(p^*_\varepsilon,\zeta^*_\varepsilon)$ of problem (3.2), and hence the following integral identity holds for any sufficiently smooth test function $\varphi(r,\theta,t)$ with $\varphi(r,\theta,T^*)=0$ $(0<T^*\le T_0)$:

$$
-\int\limits_\Omega\varepsilon p^*_\varepsilon\frac{\partial\varphi}{\partial t}\,\mathrm{d}r\mathrm{d}\theta-\int\limits_{Q_T}\mathscr{L}_*p^*_\varepsilon\varphi\,\mathrm{d}r\mathrm{d}\theta\mathrm{d}t=\int\limits_{Q_T}\Phi^\varepsilon(p^*_\varepsilon,\zeta^*_\varepsilon)\varphi\,\mathrm{d}r\mathrm{d}\theta\mathrm{d}t.\tag{6.1}
$$

Since the set of $(p^*_\varepsilon,\zeta^*_\varepsilon)$ belongs to the compact subset

$$
\{(p^*_\varepsilon,\zeta^*_\varepsilon)\mid\|p^*_\varepsilon\|_{E^{2+\alpha}(\overline{Q}_{T_0})}+\|\zeta^*_\varepsilon\|_{\widehat{E}^{2+\alpha}(\Gamma_{T_0})}\le 2C_{30}M\},
$$

one can select a convergent subsequence of it such that

$$
(p^*_{\varepsilon_k},\zeta^*_{\varepsilon_k})\ \to\ (p^*,\zeta^*)\quad\text{as}\ \ \varepsilon_k\ \to\ 0\tag{6.2}
$$

in $E^{2+\alpha}(\overline{Q}_{T_0})\times\widehat{E}^{2+\alpha}(\Gamma_{T_0})$. Passing to the limit in (6.1) along the selected subsequence, we establish that the function (p^*,ζ^*) is a solution of (3.2) with $\varepsilon=0$.

If problem (3.2) with $\varepsilon=0$ admits a unique solution on $[0,T_0^*]$ for some $T_0^*\in(0,T_0]$, then the convergence result (6.2) holds for the full sequence, so that the proof of Theorem 2.1 is completed.

Finally let us prove the uniqueness of solution (3.2) with $\varepsilon=0$. Let (p^*,ζ^*) and (p^{**},ζ^{**}) be solutions of (3.2) with $\varepsilon=0$ satisfying (5.6). Indeed, analogously to the case $\varepsilon>0$, with the help of the interpolation inequalities and (5.6) we can obtain

$$\begin{aligned}&\|(\Phi^0(p^*,\zeta^*)-\Phi^0(p^{**},\zeta^{**}),\Psi_*(\zeta^*)-\Psi_*(\zeta^{**}),\Psi_1(p^*,\zeta^*)-\Psi_1(p^{**},\zeta^{**}),0)\|_{\mathscr{H}_t^*}\\&\qquad\le(\beta''+C_{\beta''}t^{\chi''}F''(4C_{30}M))(\|p^*-p^{**}\|_{E_*^{2+\alpha}(\overline{Q}_t)}+\|\zeta^*-\zeta^{**}\|_{\hat{E}_*{}^{2+\alpha}(\Gamma_t)})\end{aligned}$$

for any $t\in(0,T_0)$ and any $\beta''>0$, where $C_{\beta''}$ is a positive constant depending on β'' non-increasingly, $\chi''>0$ is a constant depending on α (possibly), $F''(\cdot)$ is a polynomial in its argument and

$$\mathscr{H}_T^*=\underset{0}{E_*}{}^{\alpha}(\overline{Q}_T)\times\underset{0}{E_*}{}^{1+\alpha}(\Gamma_{*T})\times\underset{0}{E_*}{}^{1+\alpha}(\Gamma_T)\times\underset{0}{\check{E}_*}{}^{2+\alpha}(\Gamma_T).$$

Repeating the same arguments as in section 4 with Γ_ε, N_ε and Z_ε replaced by

$$\Gamma_0(x)=-\frac{1}{2\pi}\log|x|,\quad N_0(x_1,x_2)=\Gamma(x_1,x_2)+\Gamma(x_1,-x_2)$$

and

$$Z_0(x_1,t)=(\mathscr{F}\mathscr{L})^{-1}\left[\frac{1}{s+bd|\xi|}\right]\Bigg|_{bd=1}=\frac{1}{2\pi}\frac{t}{x_1^2+t^2},$$

respectively, we have

$$\begin{aligned}&\|p^*-p^{**}\|_{E_*^{2+\alpha}(\overline{Q}_t)}+\|\zeta^*-\zeta^{**}\|_{\widehat{E}_*^{2+\alpha}(\Gamma_t)}\\&\qquad\le C_{30}'''(\beta''+C_{\beta''}t^{\chi''}F''(4C_{30}M))(\|p^*-p^{**}\|_{E_*^{2+\alpha}(\overline{Q}_t)}+\|\zeta^*-\zeta^{**}\|_{\widehat{E}^{2+\alpha}(\Gamma_t)})\end{aligned}$$

for any $t\in(0,T_0)$. Again taking first $\beta''=1/(4C_{30}''')$, and then

$$T_0^*=\min\left\{T_0,\left(\frac{1}{4C_{30}'''C_{\beta''}F''(4C_{30}M)}\right)^{1/\chi''}\right\},$$

we conclude the uniqueness of a solution to problem (3.2) with $\varepsilon=0$ on $[0,T_0^*]$.

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