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ON SOME NEW UNIFORM ESTIMATES AND MAXIMAL THEOREMS FOR H^p SPACES

R. F. Shamoyan

Abstract: We obtain some new uniform estimates and maximal theorems in classical Hardy spaces in the unit disk related with the Bergman projection, thus extending some previously well-known inequalities on Hardy spaces.

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1. Introduction

Let $U = \{z \in C,\ |z| < 1\}$ be the unit disk in the complex plane C and T the unit circle in C. Let dm_2 be the normalized Lebesgues measure on C and U^n the unit polydisk in $C \times \cdots \times C$. Finally, let H^p be the classical analytic Hardy class in the unit disk for all positive values of p and $dm(\xi)$ the normalized Lebesgues measure on T. The goal of this short article is to extend certain classical estimates of complex function theory in the unit disk to the case of several variables, extending certain known one-dimensional results to several variables.

We use the so-called expanded Bergman projection which was recently studied by the author. We define the expanded Bergman projection for an analytic function in the unit disk as follows (see [1]):

$$(T_{n,\alpha}f)(w) = C(n,\alpha) \int\limits_U \frac{f(z)(1-|z|)^\alpha}{\prod\limits_{k=1}^{n} (1 - \langle \overline{z}, w_k \rangle)^{\frac{\alpha+2}{n}}}\, dm_2(z), \quad \alpha > -1,$$

taking the main branch of power under the integral, where by (z, w_k) we mean the conventional inner product l, $w = (w_1, \dots, w_n) \in U^n$, and $C(n, \alpha)$ is the Bergman constant from the Bergman representation formula, which plays a crucial role during the study of diagonal map (see, for example, [1–4] and the references therein).

We will now provide new estimates for this operator using, in particular, the Stein-type maximal functions from [5]. At the same time, we extend some previously known estimates.

In this paper, for any positive α, denote by $\mathscr{D}^\alpha$ the fractional derivative of a function f analytic in the unit disk:

$$\mathscr{D}^\alpha f(z) = \sum_{k=0}^{\infty} (k+1)^\alpha a_k z^k, \quad f(z) = \sum_{k=0}^{\infty} a_k z^k \quad \text{(see [1])}.$$

2. Main Result

The following theorem is the main result of this article:

Theorem 1. (a) *Let* $\Gamma_\gamma(\xi) = \{z \in U : |1 - \overline{\xi} z| < \gamma(1 - |z|)\}$, $\gamma > 1$, $\xi \in T$, *while* $\beta \in \left(0, \frac{1}{2}\right)$, $\alpha > \beta$, *and* $n = 2$. *Then*

$$\int\limits_T \Big(\sup_{z_1 \in \Gamma_\gamma(\xi)} \sup_{z_2 \in \Gamma_\gamma(\xi)} \big|\mathscr{D}^\alpha_{z_2} T_{n,0}(f)(z_1, z_2)\big| (1 - |z_1|)^{\alpha - \beta} (1 - |z_2|)^\beta \Big)^2 \, dm(\xi) \le C \|f\|^2_{H^2(U)}.$$

(b) *Let* $p > 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $t \in (-2, -1)$, $\alpha > \max\left(t + \frac{2}{q}, 0\right)$, *and* $n = 2$. *Then*

$$\sup_{z_1, z_2 \in U} |T_{n,\alpha} f(z_1, z_2)| (1 - |z_1|)^{t+2} (1 - |z_2|)^{\frac{\alpha - t}{2} - \frac{1}{q}}$$
$$\le C \bigg(\int\limits_T \sup_{z \in \Gamma_\gamma(\xi)} |f(z)| (1 - |z|)^{\frac{\alpha}{2}} \, dm(\xi) \bigg)^p.$$

Remark 1. Putting $n = 1$, $\alpha = 0$, and $\beta = 0$ in the first estimate of Theorem 1 we get the following well-known estimate for H^p spaces (by the well-known classical Bergman representation formula, $T(f) = f$ or $T(F) = F$ for these values of parameters) (see [1, Chapter 1]):

$$\int\limits_T \sup_{z \in \Gamma_\gamma(\xi)} |f(z)|^2 \, dm(\xi) \le C \|f\|^2_{H^2(U)}.$$

Putting $n = 1$, $\alpha = 0$, and $t = -2$ in the second claim of Theorem 1, we get the well-known estimate (see [6, Theorem 2.5; 1, 7])

$$\sup_{|z| < 1} |\Phi(z)| (1 - |z|)^{\frac{1}{p}} \le C \int\limits_T \sup_{w \in \Gamma_\gamma(\xi)} |\Phi(w)|^p \, dm(\xi) = \|\Phi\|^p_{H^p}.$$

Proof of Theorem 1. Let $T_{2,0}(f) = \Phi(z_1, z_2)$. Using Hölder's inequality, we obtain

$$|\Phi(z_1, z_2)| \lesssim C \int\limits_U \frac{|f(w)|}{|1 - \langle \overline{w}, z_1 \rangle| |1 - \langle \overline{w}, z_2 \rangle|} \, dm_2(w)$$
$$\le C \bigg(\int\limits_U \frac{|f(w)|^2 (1 - |w|)^{2\beta}}{|1 - \langle \overline{w}, z_1 \rangle|^2} dm_2(w) \bigg)^{\frac{1}{2}} \bigg(\int\limits_U \frac{(1 - |w|)^{-2\beta}}{|1 - \langle \overline{w}, z_2 \rangle|^2} \, dm_2(w) \bigg)^{\frac{1}{2}},$$

here we mean that there is a positive constant c such that $A < cB$ for any two positive A and B.

Since $\beta \in \left(0, \frac{1}{2}\right)$ and $\alpha > \beta$, after differentiation of positive order α of $T(f)(w)$, we get

$$\big|\mathscr{D}^\alpha_{z_2} \Phi(z_1, z_2)\big| \lesssim C \bigg(\int\limits_U \frac{|f(w)|^2 (1 - |w|)^{-2\beta}}{|1 - \langle \overline{w}, z_1 \rangle|^2} \, dm_2(w) \bigg)^{\frac{1}{2}}$$
$$\times \bigg(\int\limits_U \frac{(1 - |w|)^{2\beta}}{|1 - \langle \overline{w}, z_2 \rangle|^{2 + 2\alpha}} \, dm_2(w) \bigg)^{\frac{1}{2}},$$

and, further,

$$\sup_{z_1,z_2\in\Gamma_\gamma(\xi)}\left|\mathscr{D}^{\alpha}_{z_2}\Phi(z_1,z_2)\right|(1-|z_2|^{\alpha-\beta})(1-|z_1|^{\beta})$$
$$\le C\sup_{z_1\in\Gamma_\gamma(\xi)}\left(\int\limits_U \frac{|f(w)|^2(1-|w|)^{-2\beta}}{|1-\langle\overline{w},z_1\rangle|^2}\,dm_2(w)(1-|z_1|)^{2\beta}\right)^{\frac12}=G_1(f)\quad\text{(see [4]).}$$

Note that

$$(|1-\langle|\lambda|\overline{\xi},z\rangle|)\asymp(|1-\langle\overline{\lambda},z\rangle|),\quad z\in U,\ \lambda\in\Gamma_\beta(\xi),$$

hence,

$$G_1(f)(\xi)\lesssim C\sup_{0<r<1}\left(\int\limits_U \frac{(1-|z|)^{-2\beta}|f(z)|^2(1-r)^{2\beta}}{|1-\langle r\overline{\xi},z\rangle|^2}dm_2(z)\right)^{\frac12}=\widetilde{G_1}(f,\xi,\beta).$$

Obviously, for $\gamma\in(1,2-2\beta)$ and $\beta\in\left(0,\frac12\right)$, we have

$$\widetilde{G_1}(f,\xi,\beta)\lesssim C\sup_{0<r<1}\left(\int\limits_T \frac{(1-r)^{\gamma-1}|f(r\xi)|^2}{|1-\langle r\xi,\varphi\rangle|^\gamma}\,dm(\xi)\right)^{\frac12}.$$

Therefore, to get what we need, it suffices to use the estimates for the Stein-type maximal functions [5]

$$\left\|\sup_{0<r<1}\left(\int\limits_T \frac{(1-r)^{\alpha-1}|f(r\varphi)|^p}{|1-\langle r\overline{\varphi},\xi\rangle|^\alpha}\,d\varphi\right)^{\frac1p}\right\|_{L^p}\le C\|f\|_{H^p},$$

where $f\in H^p$, $p>1$, $\beta\in\left(0,\frac1p\right)$, and $\alpha\in(1,2-\beta p)$. The first estimate is proven.

Let us prove the second estimate. First, we have the following chain of well-known estimates (see, for example, [7]):

$$\int\limits_U d\mu(z)\lesssim C\int\limits_T\int\limits_{\Gamma_t(\xi)}\frac{d\mu(z)}{1-|z|}\,dm(\xi),\tag{1}$$

$$\int\limits_T |M_{H-L}f(\xi)|^p\,d\xi\le C\int\limits_T |f(\xi)|^p\,d\xi,\quad p>1,\tag{2}$$

where M_{H-L} is the classical maximal Hardy–Littlewood operator;

$$\int\limits_U |f(z)|^{\widetilde{p}}\,dm_2(z)\lesssim C\int\limits_T(\sup_{z\in\Gamma_\gamma(\xi)}|f(z)|)^{\widetilde{p}}C(\mu)(\xi)\,d\xi,\tag{3}$$

where μ is a positive Borel measure, $0<\widetilde{p}<\infty$, f is measurable in U, and, as usual,

$$C(\mu)(\xi)=\sup_{\xi\in I}\frac{1}{|I|}\int\limits_{\Delta I}d\mu(\xi),\quad \Delta I=\{z=r\xi,\ \xi\in I,\ 1-|z|<r<1\},\ I\subset\ T.$$

Using (1), we get

$$|\Phi(z_1,z_2)|\lesssim C(\alpha)\int\limits_U \frac{|f(w)|(1-|w|^\alpha)}{(1-\langle z_1,\overline{w}\rangle)^{\frac{\alpha+2}{2}}(1-\langle z_2,\overline{w}\rangle)^{\frac{\alpha+2}{2}}}\,dm_2(w),$$

where $z_1, z_2 \in U$ and $C(\alpha)$ is the Bergman projection constant. Now, using (1) and applying Hölder's inequality twice, we obtain

$$|\Phi(z_1,z_2)| \lesssim C\biggl(\int\limits_T \biggl(\int\limits_{\Gamma_\alpha(\xi)} \frac{|f(w)|^2(1-|w|)^{2\alpha-t}dm_2(w)}{(1-|w|)^2|1-\langle z_1,\overline{w}\rangle|^{\alpha+2}}\biggr)^{\frac{p}{2}} d\xi\biggr)^{\frac{1}{p}}$$

$$\times\biggl(\int\limits_T \biggl(\int\limits_{\Gamma_\alpha(\xi)} \frac{(1-|w|)^t dm_2(w)}{|1-\langle z_2,\overline{w}\rangle|^{\alpha+2}}\biggr)^{\frac{q}{2}} dm(\xi)\biggr)^{\frac{1}{q}} \lesssim B(f)(1-|z_2|)^{-(\frac{\alpha-t}{2}-\frac{1}{q})},$$

where $p>2$, $\alpha > t+\frac{2}{q}$, $t\in(-2,-1)$, $\alpha>\frac{t}{2}$, $\frac{1}{p}+\frac{1}{q}=1$.

Using Fubini's theorem and also the well-known duality argument, we have

$$B(f) = \sup_{\|\varphi\|_{L^{(\frac{p}{2})'}}} \int\limits_T \int\limits_{\Gamma_\alpha(\xi)} \frac{|f(w)|^2(1-|w|)^{2\alpha-t}dm_2(w)}{(1-|w|)^2|1-\langle z_1,\overline{w}\rangle|^{\alpha+2}}|\psi(\xi)|\,dm(\xi)$$

$$= \sup_{\|\varphi\|_{L^{(\frac{p}{2})'}}} \int\limits_U \frac{|f(w)|^2(1-|w|)^{2\alpha-t}}{|1-\langle z_1,\overline{w}\rangle|^{\alpha+2}} \int\limits_T |\psi(\xi)|\chi_{\Gamma_\alpha(\xi)}(z)\,d\xi \frac{dm_2(w)}{(1-|w|)^2}.$$

Hence, using (3) and the estimate

$$\sup_{z\in\Gamma_\eta} \frac{1}{1-|z|}\int\limits_T |\psi(\xi)|\chi_{\Gamma_\tau(\xi)}(z)\,dm(\xi) \le CM_{H-L}(\varphi)(\xi) \quad \text{(see [4])},$$

we obtain ($\widetilde{f} = f(1-|w|)^{\frac{\alpha}{2}}$)

$$B(f) \lesssim \sup_\varphi \int\limits_T (A_\infty(\widetilde{f})(\xi))^2 M_{H-L}(\varphi)(\xi) C\biggl(\frac{(1-|w|)^{\alpha-t-1}}{|1-\langle z,w\rangle|^{\alpha+2}}\biggr)(\xi)\,d\xi$$

$$\lesssim \sup_\varphi \int\limits_T (A_\infty(\widetilde{f})(\xi))^2 M_{H-L}(\varphi)(\xi)\,d\xi \sup_{\widetilde{w}\in U}\int\limits_U \frac{(1-|w|)^{\alpha-t-1}(1-|\widetilde{w}|)^N\,dm_2(w)}{|1-\langle z_1,\widetilde{w}\rangle|^{\alpha+2}|1-\langle \widetilde{w},w\rangle|^{N+1}},$$

where M_{H-L} is the maximal Hardy–Littlewood function. We used the fact that

$$\|C(F)\|_{L^\infty} = \sup_{\widetilde{w}\in U}\int\limits_U \frac{|F(z)|dm_2(z)}{|1-\langle\widetilde{w},z\rangle|^N}(1-|\widetilde{w}|)^{N-1}, \quad N>1.$$

From the last estimate, Hölder's inequality and (2), we finally have

$$|\Phi(z_1,z_2)|(1-|z_1|)^{t+2}(1-|z_2|)^{\frac{\alpha-t}{2}-\frac{1}{q}} \le C\|\widetilde{f}\|_{L^p},$$

$t\in(-2,-1)$, $p>2$. Theorem 1 is proven. □

We similarly show the following results for the multifunctional case; some estimates are valid even in the unit ball.

We assume both parameters α and β to be positive in the following theorems.

Theorem 2. *Let $\alpha+\beta<\frac{1}{2}$, $\alpha>\beta$, and $\beta<\frac{1}{2}$. Then*

$$\int\limits_T \Big(\sup_{z_1\in\Gamma_\gamma(\xi)}\sup_{z_2\in\Gamma_\gamma(\xi)}\big|\mathscr{D}^{\alpha}_{z_2}(T_{2,0}\widetilde{f})(z_1,z_2)\big|(1-|z_1|)^{\alpha-\beta}(1-|z_2|)^{\beta+\alpha}\Big)^2\,d\xi$$
$$\le C\|f_1\|^2_{H^2(U)}\|f_2\|^2_{H^{\frac{m-1}{\alpha}}}\cdots\|f_m\|^2_{H^{\frac{m-1}{\alpha}}},$$

where $\widetilde{f}=\prod\limits_{j=1}^{m}f_j$, $m\ge 2$, $m\in\mathbb{N}$.

Theorem 3. *Let $\alpha+\beta<\frac{1}{2}$, $\alpha>\beta$, and $\beta<\frac{1}{2}$. Then*

$$\int\limits_T \Big(\sup_{z_1\in\Gamma_\gamma(\xi)}\sup_{z_2\in\Gamma_\gamma(\xi)}\big|\mathscr{D}^{\alpha}_{z_2}(T_{2,0})(\widetilde{f})(z_1,z_2)\big|(1-|z_1|)^{\alpha-\beta}(1-|z_2|)^{\beta+\alpha}\Big)^2\,dm(\xi)$$
$$\le \widetilde{C}\|f_1\|^2_{H^2(U)}\|f_2\|^2_{\frac{\alpha}{m-1}}\cdots\|f_m\|^2_{\frac{\alpha}{m-1}},$$

where $\|f\|_t=(\sup\limits_{z\in U})|f(z)|(1-|z|)^t$, $t\ge 0$, $\widetilde{f}=\prod\limits_{j=1}^{m}f_j$, $m\ge 2$.

We will show these new results in the second part of this paper.

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Shamoyan Romi F.
Department of Mathematics,
Bryansk State Technical University,
7 50-letiya Oktyabrya Boulevard, Bryansk 241035, Russia
`rsham@mail`