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THE SHARP BOUND OF THE GENERALIZED ZALCMAN CONJECTURE FOR INITIAL COEFFICIENT AND THE CERTAIN SECOND HANKEL DETERMINANTS OF k^{th}–ROOT TRANSFORMATION FOR A SUBCLASS OF ANALYTIC FUNCTIONS

K. S. Kumar, B. Rath, D. V. Krishna, and G. K. S. Viswanadh

Abstract: We obtain sharp bounds in the generalized Zalcman conjecture for the initial coefficients and the second Hankel determinants $H_{2,1,k}(f)$, $H_{2,2,k}(f)$ for the k^{th}-root transformation to the subclass of analytic functions.

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1. Introduction

Denote by $\mathscr{H}$ the family of all analytic functions in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathscr{A}$ be the subfamily of functions f normailized by $f(0) = f'(0) - 1 = 0$, i.e, of the type

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_1 := 1, \tag{1.1}$$

and $\mathscr{S}$ be the subfamily of $\mathscr{A}$ possessing univalent (schlicht) mappings.

Let k be a positive integer. A domain $U \in \mathbb{C}$ is said to be *k-fold symmetric* if a rotation of U about the origin through an angle $\frac{2\pi}{k}$ carries U to itself. A function h is said to be *k-fold symmetric in* $\mathbb{D}$ if $h(e^{\frac{2\pi i}{k}} z) = e^{\frac{2\pi i}{k}} h(z)$ for every $z \in \mathbb{D}$. If h is regular and k-fold symmetric in $\mathbb{D}$, then

$$h(z) = b_1 z + b_{k+1} z^{k+1} + b_{2k+1} z^{2k+1} + \cdots . \tag{1.2}$$

Conversely, if h is given by (1.2), then h is k-fold symmetric inside the circle of convergence of the series (see [1]). The k^{th}-root transform for the mapping f in (1.1) is

$$G(z) := [f(z^k)]^{\frac{1}{k}} = z + \sum_{n=1}^{\infty} b_{kn+1} z^{kn+1}. \tag{1.3}$$

Vamshee et al. [2] introduced and interpreted the concept of Hankel determinant for $G(z)$ for f in (1.1), with $q, t, k \in \mathbb{N} = \{1, 2, 3, \dots\}$, as

$$H_{q,t,k}(f) = \begin{vmatrix} b_{k(t-1)+1} & b_{kt+1} & \cdots & b_{k(t+q-2)+1} \\ b_{kt+1} & b_{k(t+1)+1} & \cdots & b_{k(t+q-1)+1} \\ \vdots & \vdots & \vdots & \vdots \\ b_{k(t+q-2)+1} & b_{k(t+q-1)+1} & \cdots & b_{k[t+2(q-1)-1]+1} \end{vmatrix} \quad (b_1 = 1). \tag{1.4}$$

In particular, if $k = 1$ in (1.4), then it reduces to $H_{q,t,1}(f) = H_{q,t}(f)$, the Hankel determinant defined by Pommerenke [3] for the function f given in (1.1).

The hypergeometric function ${}_2F_1(a, b; c; z)$ is defined for $|z| \leq 1$ by the power series

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} = 1 + \frac{ab}{c} \frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{z^2}{2!} + \cdots .$$

It is undefined (or infinite) if c equals a non-positive integer. Here $(q)_n$ is the (rising) *Pochhammer symbol*, which is defined as follows:

$$(q)_n = \begin{cases} 1, & n = 0, \\ q(q+1)\cdots(q+n-1), & n > 0. \end{cases}$$

Ali et al. [4] derived exact estimates for $|b_{2k+1} - \mu b_{k+1}^2|$, the generalized Fekete–Szegö functional related to the function $G(z)$, while Vamshee et al. [5] studied certain second Hankel determinants when f is a member of specific subfamilies of S.

S. Owa [6, 7] studied the class $\mathscr{I}_\beta$ with anaytic conditions

$$\mathrm{Re}\left\{(1-\beta)\frac{f(z)}{z} + \beta f'(z)\right\} > 0, \quad \beta \geq 0,$$

which was also studied by H. Saitoh [8].

For our study in this paper we consider second Hankel determinants $H_{2,1,k}(f)$ and $H_{2,2,k}(f)$ and generalized Zalcman for initial coefficient of k^{th}-root transformation for $\mathscr{I}_\beta$.

Denote by $\mathscr{P}$ the collection of all functions g called Carathéodory functions [9], of the form

$$g(z) = 1 + \sum_{t=1}^{\infty} c_t z^t, \tag{1.5}$$

holomorphic in $\mathbb{D}$ and such that $\mathrm{Re}\, g(z) > 0$ for $z \in \mathbb{D}$. The classes $\mathscr{I}_\beta$ and $\mathscr{P}$ are invariant under the rotations by Carathéodory Theorem (see [10, Vol. I, p. 80, Theorem 3]).

For the proof of our main result we need the following lemmas, which contain the well-known formulas for c_2 (e.g., [11, p. 166]) and for c_3 due to Libera and Zlotkiewicz [12–14].

Lemma 1.1 [11]. *If $g \in \mathscr{P}$, then $|c_t| \leq 2$ for $t \in \mathbb{N}$; the equality is attained for the function $g(z) = \frac{1+z}{1-z}$, $z \in \mathbb{D}$.*

Lemma 1.2 [15, 16]. *If $g \in \mathscr{P}$, then $|c_i - \mu c_j c_{i-j}| \le 2$ for $i, j \in \mathbb{N}$, $i > j$, and $\mu \in [0,1]$, which is same as $|c_{n+k} - \mu c_n c_k| \le 2$ for $n, k \in \mathbb{N}$ with $\mu \in [0,1]$.*

Lemma 1.3 [17]. *If $g \in \mathscr{P}$, then*

$$\left|Jc_1^3 - Kc_1c_2 + Lc_3\right| \le 2(|J| + |K - 2J| + |J - K + L|).$$

Lemma 1.4 [18]. *If $g \in \mathscr{P}$, then*

$$2c_2 = c_1^2 + t\zeta, \quad 4c_3 = c_1^3 + 2c_1t\zeta - c_1t\zeta^2 + 2t(1 - |\zeta|^2)\eta,$$

while

$$8c_4 = c_1^4 + 3c_1^2t\zeta + \left(4 - 3c_1^2\right)t\zeta^2 + c_1^2t\zeta^3 + 4t(1 - |\zeta|^2)(1 - |\eta|^2)\xi \\ + 4t(1 - |\zeta|^2)(c_1\eta - c_1\zeta\eta - \overline{\zeta}\eta^2),$$

where $t := 4 - c_1^2$, for some ζ, η and ξ such that $|\zeta| \le 1$, $|\eta| \le 1$, and $|\xi| \le 1$.

Result 1.1. *Suppose that $\Psi : [0,2] \to \mathbb{R}$, $k \in \mathbb{N}$, $\beta \ge 0$, and $\Psi(c)$ is defined as*

$$\Psi(c) = \frac{1}{12}c^2\left(\frac{k^2 - 1}{(\beta+1)^4k^4} - \frac{6\beta^2}{(3\beta^2 + 4\beta + 1)(2\beta k + k)^2}\right) \\ - \frac{(4\beta + 1)}{(3\beta^2 + 4\beta + 1)(2\beta k + k)^2}.$$

Then

$$\Psi(c) \le 0.$$

Proof. We can easily see that

$$12(\beta+1)^4(2\beta+1)^2(3\beta+1)k^4\Psi(c) = -6\beta^2(\beta+1)^3c^2k^2 - 12(4\beta+1)(\beta+1)^3k^2 \\ + (2\beta+1)^2(3\beta+1)(k^2-1)c^2 = -[c^2(12\beta^3 + 16\beta^2 + 7\beta + 6\beta^5k^2 + 18\beta^4k^2 + 6\beta^3k^2 + 1) \\ + (2\beta+1)(5\beta+1)(12 - c^2)k^2 + 12\beta^2(4\beta^2 + 13\beta + 5)k^2] \le 0;$$

therefore, $\Psi(c) \le 0$. $\square$

2. Main Results

Theorem 2.1. *If $f \in \mathscr{I}_\beta$, $\beta \ge 0$, then*

$$|H_{2,1,k}(f)| \le \frac{2}{(2\beta+1)k},$$

and the result is sharp for $f_0(z) := 2z{}_2F_1\left(1, \frac{1}{2\beta}; \frac{1}{2}\left(2 + \frac{1}{\beta}\right); z^2\right) - z$.

Proof. For $f \in \mathscr{I}_\beta$, there exists a holomorphic function $g \in \mathscr{P}$ such that

$$(1-\beta)\frac{f(z)}{z} + \beta f'(z) = g(z), \quad z \in \mathbb{D}. \tag{2.1}$$

Substitute the values for f, f' and g in (2.1), then

$$a_{n+1} = \frac{c_n}{1 + \beta n}, \quad n \in \mathbb{N}. \tag{2.2}$$

For the mapping f in (1.1), a simple calculation gives

$$\begin{aligned}[f(z^k)]^{\frac{1}{k}} &= \left[z^k + \sum_{n=2}^{\infty} a_n z^{nk}\right]^{\frac{1}{k}} \\ &= \left[z + \frac{1}{k}a_2 z^{k+1} + \left\{\frac{1}{k}a_3 + \frac{(1-k)}{2k^2}a_2^2\right\} z^{2k+1}\right. \\ &\quad + \left\{\frac{1}{k}a_4 + \frac{(1-k)}{k^2}a_2a_3 + \frac{(1-k)(1-2k)}{6k^3}a_2^3\right\} z^{3k+1} \\ &\quad + \left\{\frac{1}{k}a_5 + \frac{(1-k)}{2k^2}(a_3^2 + 2a_2a_4) + \frac{(1-k)(1-2k)}{2k^3}a_2^2a_3\right. \\ &\quad \left.\left. + \frac{(1-k)(1-2k)(1-3k)}{24k^4}a_2^4\right\} z^{4k+1} + \cdots\right]. \end{aligned} \tag{2.3}$$

Comparing the coefficients of z^{k+1}, z^{2k+1}, z^{3k+1}, and z^{4k+1} in (1.2) and (2.3), we get

$$\begin{aligned} b_{k+1} &= \frac{1}{k}a_2, \\ b_{2k+1} &= \frac{1}{k}a_3 + \frac{(1-k)}{2k^2}a_2^2, \\ b_{3k+1} &= \left[\frac{1}{k}a_4 + \frac{(1-k)}{k^2}a_2a_3 + \frac{(1-k)(1-2k)}{6k^3}a_2^3\right], \end{aligned} \tag{2.4}$$

$$\begin{aligned} b_{4k+1} = \Bigg[\frac{1}{k}a_5 &+ \frac{(1-k)}{2k^2}\left(a_3^2 + 2a_2a_4\right) + \frac{(1-k)(1-2k)}{2k^3}a_2^2a_3 \\ &+ \frac{(1-k)(1-2k)(1-3k)}{24k^4}a_2^4\Bigg]. \end{aligned}$$

From (2.2) and (2.4), we obtain

$$\begin{aligned} b_{k+1} &= \frac{c_1}{(\beta+1)k}, \quad b_{2k+1} = \frac{c_1^2(1-k)}{2(\beta+1)^2k^2} + \frac{c_2}{(2\beta+1)k}, \\ b_{3k+1} &= \frac{c_1^3(1-k)(1-2k)}{6(\beta+1)^3k^3} + \frac{c_2c_1(1-k)}{(\beta+1)(2\beta+1)k^2} + \frac{c_3}{(3\beta+1)k}, \end{aligned} \tag{2.5}$$

$$\begin{aligned} b_{4k+1} = \frac{(1-k)(1-2k)(1-3k)c_1^4}{24(\beta+1)^4k^4} &+ \frac{(1-k)(1-2k)c_1^2c_2}{2(1+\beta)^2(1+2\beta)k^3} + \frac{(1-k)c_2^2}{2(2\beta+1)^2k^2} \\ &+ \frac{(1-k)c_1c_3}{2(\beta+1)(3\beta+1)k^2} + \frac{c_4}{(4\beta+1)k}. \end{aligned}$$

Now, for $q = 2$ and $t = 1$ in (1.4), we have

$$H_{2,1,k}(f) = \begin{vmatrix} 1 & b_{k+1} \\ b_{k+1} & b_{2k+1} \end{vmatrix}. \tag{2.6}$$

Using the values of b_j $(j = k+1, 2k+1)$ from (2.5) in (2.6), we get

$$H_{2,1,k}(f) = \frac{c_2}{(2\beta+1)k} - \frac{(k+1)c_1^2}{2(\beta+1)^2k^2}. \tag{2.7}$$

Taking modulus on both side and applying Lemma 1.2, we have

$$|H_{2,1,k}(f)| = \frac{1}{(2\beta+1)k}\left|c_2 - \frac{(k+1)(2\beta+1)c_1^2}{2k(\beta+1)^2}\right| \leq \frac{2}{(2\beta+1)k}. \tag{2.8}$$

For $f_0(z)$ we obtain $a_1 = 1$, $a_2 = 0$, and $a_3 = 2/(1+2\beta)$; further, $b_{k+1} = 0$ and $b_{k+1} = 2/(1+2\beta)k$, whence the result follows. $\square$

Theorem 2.2. *If $f \in \mathscr{I}_\beta$ and $\beta \geq 0$, then*

$$|H_{2,2,k}(f)| \leq \frac{4}{(2\beta k+k)^2},$$

and the result is sharp for $f_0(z)$ given in Theorem 2.1.

Proof. Now, for $q = 2$ and $t = 2$ in (1.4), we get

$$H_{2,2,k}(f) = \begin{vmatrix} b_{k+1} & b_{2k+1} \\ b_{2k+1} & b_{3k+1} \end{vmatrix}. \tag{2.9}$$

Putting the calculated values of b_{jk+1}, for $j \in \{1,2,3\}$ from (2.5) into (2.9), we get

$$H_{2,2,k}(f) = \frac{c_3c_1}{(3\beta^2+4\beta+1)k^2} + \frac{c_1^4(k^2-1)}{12(\beta+1)^4k^4} - \frac{c_2^2}{(2\beta k+k)^2}. \tag{2.10}$$

Substituting the values of c_2 and c_3 from Lemma 1.4 on the right-hand side of (2.10), it simplifies to

$$\begin{aligned} H_{2,2,k}(f) = {} & \frac{c_1^4(k^2-1)}{12(\beta+1)^4k^4} + \frac{\beta^2c_1^4}{4(3\beta^2+4\beta+1)(2\beta k+k)^2} \\ & + (4-c_1^2)\bigg(-\frac{c_1^2\zeta^2}{4(3\beta^2+4\beta+1)k^2} + \frac{c_1(1-|\zeta|^2)\eta}{2(3\beta^2+4\beta+1)k^2} \\ & \qquad - \frac{(4-c_1^2)\zeta^2}{4(2\beta k+k)^2} + \frac{\beta^2c_1^2\zeta}{2(3\beta^2+4\beta+1)(2\beta k+k)^2}\bigg). \end{aligned} \tag{2.11}$$

Taking modulus on both sides and then applying the triangle inequality in the above expression, while using $c_1 := c \in [0,2]$, $|\zeta| := x \in [0,1]$, and $|\eta| \leq 1$, we have

$$\begin{aligned} |H_{2,2,k}(f)| \leq {} & \frac{c^4(k^2-1)}{12(\beta+1)^4k^4} + \frac{\beta^2c^4}{4(3\beta^2+4\beta+1)(2\beta k+k)^2} \\ & + (4-c^2)\bigg(\frac{c^2x^2}{4(3\beta^2+4\beta+1)k^2} + \frac{c(1-x^2)}{2(3\beta^2+4\beta+1)k^2} \\ & \qquad + \frac{(4-c^2)x^2}{4(2\beta k+k)^2} + \frac{\beta^2c^2x}{2(3\beta^2+4\beta+1)(2\beta k+k)^2}\bigg), \end{aligned}$$

which is equivalent to

$$\begin{aligned} |H_{2,2,k}(f)| \leq {} & \frac{c^4(k^2-1)}{12(\beta+1)^4k^4} + \frac{\beta^2c^4}{4(3\beta^2+4\beta+1)(2\beta k+k)^2} \\ & + (4-c^2)\bigg(\frac{c}{2(3\beta^2+4\beta+1)k^2} + \frac{\beta^2c^2x}{2(3\beta^2+4\beta+1)(2\beta k+k)^2} \end{aligned}$$

$$+\frac{(2-c)x^2(6\beta^2+8\beta+\beta^2(-c)+2)}{4(3\beta^2+4\beta+1)(2\beta k+k)^2}\Bigg).$$

For $c \in [0,2]$ and $\beta \geq 0$ all terms in RHS are positive, thus,

$$\begin{aligned}|H_{2,2,k}(f)| &\leq \frac{c^4(k^2-1)}{12(\beta+1)^4k^4}+\frac{\beta^2c^4}{4(3\beta^2+4\beta+1)(2\beta k+k)^2}\\ &+(4-c^2)\Bigg(\frac{c}{2(3\beta^2+4\beta+1)k^2}+\frac{\beta^2c^2}{2(3\beta^2+4\beta+1)(2\beta k+k)^2}\\ &+\frac{(2-c)(6\beta^2+8\beta+\beta^2(-c)+2)}{4(3\beta^2+4\beta+1)(2\beta k+k)^2}\Bigg)\\ &=\frac{1}{12}c^4\left(\frac{k^2-1}{(\beta+1)^4k^4}-\frac{6\beta^2}{(3\beta^2+4\beta+1)(2\beta k+k)^2}\right)\end{aligned}$$

$$\frac{(-4\beta-1)c^2}{(3\beta^2+4\beta+1)(2\beta k+k)^2}+\frac{4}{(2\beta k+k)^2}=\Psi(c)c^2+\frac{4}{(2\beta k+k)^2}\leq\frac{4}{(2\beta k+k)^2},$$

since $\Psi(c) \leq 0$ by Result 1.1.

For $f_0(z)$ we obtain $a_1 = 1$, $a_2 = a_4 = 0$, and $a_3 = 2/(1+2\beta)$, then, $b_{k+1} = b_{3k+1} = 0$ and $b_{k+1} = 2/(1+2\beta)k$, whence follows the result. □

Theorem 2.3. *If $f \in \mathscr{I}_\beta$ and $\beta \geq 0$, then*

$$|b_{3k+1}-b_{k+1}b_{2k+1}| \leq \frac{2}{(3\beta+1)k},$$

and the result is sharp for $f_1(z) := 2z{}_2F_1\big(1, \frac{1}{3\beta}; \frac{1}{3}(3+\frac{1}{\beta}); z^3\big) - z$.

Proof. Using the values of b_{jk+1} for $j \in \{1,2,3\}$ from (2.5) in the expression $b_{3k+1} - b_{k+1}b_{2k+1}$, we get

$$b_{3k+1}-b_{k+1}b_{2k+1}=\frac{c_1^3(k^2-1)}{3(\beta+1)^3k^3}-\frac{c_2c_1}{(\beta+1)(2\beta+1)k}+\frac{c_3}{(3\beta+1)k}. \tag{2.12}$$

Taking modulus on both sides and then applying Lemma 1.3, we have

$$\begin{aligned}|b_{3k+1}-b_{k+1}b_{2k+1}| &\leq 2\Bigg[\left|\frac{k^2-1}{3(\beta+1)^3k^3}\right|+\left|\frac{1}{(\beta+1)(2\beta+1)k}-\frac{2(k^2-1)}{3(\beta+1)^3k^3}\right|\\ &+\left|\frac{k^2-1}{3(\beta+1)^3k^3}-\frac{1}{(\beta+1)(2\beta+1)k}+\frac{1}{(3\beta+1)k}\right|\Bigg]\\ &=2\Bigg[\frac{k^2-1}{3(\beta+1)^3k^3}+\frac{4\beta+3\beta^2k^2+2\beta k^2+k^2+2}{3(\beta+1)^3(2\beta+1)k^3}\\ &+\frac{-6\beta^2-5\beta+6\beta^4k^2+12\beta^3k^2+12\beta^2k^2+5\beta k^2+k^2-1}{3(\beta+1)^3(2\beta+1)(3\beta+1)k^3}\Bigg]=\frac{2}{(3\beta+1)k}.\end{aligned}$$

For $f_1(z)$ we obtain $a_1 = 1$, $a_2 = a_3 = 0$, and $a_4 = 2/(1+3\beta)$, while $b_{k+1} = b_{2k+1} = 0$ and $b_{3k+1} = 2/(1+3\beta)k$, whence follows the required. □

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K. Sanjay Kumar, Biswajit Rath (corresponding author),
D. Vamshee Krishna, and G. K. Surya Viswanadh
Department of Mathematics,
GITAM School of Science, GITAM
Visakhapatnam-530 045, A. P., India
skarri9@gitam.in, brath@gitam.edu, vamsheekrishna1972@gmail.com,
svsu06@gmail.com