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STRUCTURED PSEUDOSPECTRUM AND STRUCTURED ESSENTIAL PSEUDOSPECTRUM OF CLOSED LINEAR OPERATOR PENCILS ON ULTRAMETRIC BANACH SPACES

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Abstract: We introduce and study the structured pseudospectrum and the essential pseudospectrum of closed linear operator pencils on ultrametric Banach spaces. We establish a characterization of the structured pseudospectrum of closed linear operator pencils and relationship between the structured pseudospectrum and the structured pseudospectrum of closed linear operator pencils on ultrametric Banach spaces. Many characterizations of structured essential pseudospectra of operators, such as the structured essential pseudospectrum of closed linear operator pencils, is invariant under perturbation of completely continuous linear operators on ultrametric Banach spaces over $\mathbb{Q}_p$. Finally, we give some illustrative examples.

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1. Introduction and Preliminaries

In the classical functional analysis, Trefethen [1] studied and developed the pseudospectrum of matrices and bounded linear operators. Davies [2] introduced and studied the structured pseudospectrum of a closed linear operator S on complex Banach space $\mathscr{E}$ over $\mathbb{C}$ and he gave a characterization of the structured pseudospectrum of S. For more details, see [2].

In ultrametric operator theory, the pseudospectra of linear operators were extended and studied by the authors in [3] and they characterized the pseudospectrum of linear operators and the essential pseudospectrum of closed linear operators. In [4], Ammar et al. introduced and studied the condition pseudospectrum of bounded linear operators on ultrametric Banach spaces. They established a relationship between the condition pseudospectrum and the pseudospectrum and they proved some properties of the essential condition pseudospectrum. Recently, El Amrani et al. [5] studied the pseudospectrum of ultrametric matrices, the condition pseudospectrum of ultrametric matrices and the pseudospectrum of ultrametric matrix pencils. They showed some results about them and they gave some illustrative examples. Furthermore, the trace pseudospectrum of ultrametric matrix pencils, the determinant pseudospectrum of ultrametric matrix pencils and the condition pseudospectrum of ultrametric operator pencils were studied by several authors. For more details, we refer to [5–8].

The eigenvalue problem is one of interesting problems in ultrametric operator theory. It played an important role in many parts of ultrametric applied mathematics and physics including matrix theory, ultrametric pseudo-differential equations, control theory and ultrametric quantum mechanics. For more details, see [3, 4, 9, 10]. This work is motivated by many studies of ultrametric spectral theory and perturbation theory of linear operators. For more details, we refer to [5–8, 11, 12].

Throughout this paper, $\mathbb{K}$ is a complete ultrametric field with a non-trivial valuation $|\cdot|$, $\mathscr{E}$ is an ultrametric Banach space over $\mathbb{K}$, $\mathscr{B}(\mathscr{E})$ denotes the collection of all bounded linear operators on $\mathscr{E}$, $\mathscr{C}(\mathscr{E})$ is the set of all closed, densely defined linear operators on $\mathscr{E}$, $\mathscr{E}^* = \mathscr{B}(\mathscr{E}, \mathbb{K})$ is the dual space of $\mathscr{E}$ and $\mathbb{Q}_p$ is the field of p-adic numbers. For more details, we refer to [13, 14]. For $S \in \mathscr{C}(\mathscr{E})$, $D(S)$, $N(S)$, $R(S)$, $\rho(S)$, $\sigma(S)$ and $\sigma_e(S)$ are the domain, the kernel, the range, the resolvent set, the spectrum and the essential spectrum of S respectively. Recall that an unbounded linear operator $S : D(S) \subset \mathscr{E} \longrightarrow \mathscr{E}$ is said to be closed, densely defined if S is closed and $D(S)$ is dense in $\mathscr{E}$. For more details on closed, densely defined linear operators, see [11, 15]. We begin with the following preliminaries.

Definition 1 [13]. A field $\mathbb{K}$ is said to be *ultrametric* if it is endowed with an absolute value $|\cdot| : \mathbb{K} \to \mathbb{R}^+$ such that

(i) $|\alpha| = 0$ if, and only if, $\alpha = 0$;
(ii) For all $\alpha, \mu \in \mathbb{K}$, $|\alpha\mu| = |\alpha||\mu|$;
(iii) For each $\alpha, \mu \in \mathbb{K}$, $|\alpha + \mu| \leq \max\{|\alpha|, |\mu|\}$.

From now, we assume that $\mathbb{K} = (\mathbb{K}, |\cdot|)$ is a complete ultrametric valued field.

Definition 2 [13]. An ultrametric field $\mathbb{K}$ is said to be *spherically complete* if each decreasing sequence of closed balls $(B_n)_n$ has nonempty intersection.

Definition 3 [13]. Let $\mathscr{E}$ be a vector space over $\mathbb{K}$. A function $\|\cdot\| : \mathscr{E} \to \mathbb{R}_+$ is called an *ultrametric norm* if:

(i) For all $u \in \mathscr{E}$, $\|u\| = 0$ if and only if $u = 0$;
(ii) For all $u \in \mathscr{E}$ and $\lambda \in \mathbb{K}$, $\|\lambda u\| = |\lambda|\|u\|$;
(iii) For any $u, v \in \mathscr{E}$, $\|u + v\| \leq \max(\|u\|, \|v\|)$.

Definition 4 [13]. An ultrametric Banach space is a *complete ultrametric normed space.*

Ingleton [14] proved the following theorem.

Theorem 1 [14]. *Assume that $\mathbb{K}$ is spherically complete. Let $\mathscr{E}$ be an ultrametric Banach space over $\mathbb{K}$. For all $u \in \mathscr{E}\backslash\{0\}$, there is $u^* \in \mathscr{E}^*$ such that $u^*(u) = 1$ and $\|u^*\| = \|u\|^{-1}$.*

Definition 5 [13]. Let $\omega = (\omega_i)_i$ be a sequence of $\mathbb{K}\backslash\{0\}$. We define $\mathscr{E}_\omega$ by

$$\mathscr{E}_\omega = \{u = (u_i)_i : \forall i \in \mathbb{N},\ u_i \in \mathbb{K} \text{ and } \lim_{i\to\infty} |\omega_i|^{\frac{1}{2}}|u_i| = 0\}.$$

On $\mathscr{E}_\omega$, we define

$$\forall u \in \mathscr{E}_\omega : u = (u_i)_i,\ \|u\| = \sup_{i\in\mathbb{N}}(|\omega_i|^{\frac{1}{2}}|u_i|).$$

Then $(\mathscr{E}_\omega, \|\cdot\|)$ is an ultrametric Banach space.

Remark 1 [13]. The orthogonal basis $\{e_i,\ i \in \mathbb{N}\}$ is called the *canonical basis* of $\mathscr{E}_\omega$ where $e_i = (\delta_{i,j})_{j\in\mathbb{N}}$ and $\delta_{i,j}$ is the Kronecker symbol. For each $i \in \mathbb{N}$, $\|e_i\| = |\omega_i|^{\frac{1}{2}}$.

Definition 6 [13]. Let $S \in \mathscr{C}(\mathscr{E})$, S is called an upper semi-Fredholm operator if $\alpha(S) = \dim N(S)$ is finite and $R(S)$ is closed.

The collection of each upper semi-Fredholm operators on $\mathscr{E}$ is denoted by $\Phi_+(\mathscr{E})$.

Definition 7 [13]. Let $S \in \mathscr{C}(\mathscr{E})$, S is said to be a *lower semi-Fredholm operator* if $\beta(S) = \dim(\mathscr{E}/R(S))$ is finite.

The collection of each lower semi-Fredholm operators on $\mathscr{E}$ is denoted by $\Phi_-(\mathscr{E})$.

The collection of each closed Fredholm operators on $\mathscr{E}$ is

$$\Phi(\mathscr{E}) = \Phi_+(\mathscr{E}) \cap \Phi_-(\mathscr{E}).$$

For more details on closed Fredholm operators, see [13].

Definition 8 [14]. Let $S \in \mathscr{B}(\mathscr{E})$, S is said to be an operator of finite rank if $\dim R(S)$ is finite.

The set of all finite rank operators on $\mathscr{E}$ will be denoted by $\mathscr{F}_0(\mathscr{E})$.

Definition 9 [13]. Let $\mathscr{E}$ be an ultrametric Banach space and let $S \in \mathscr{B}(\mathscr{E})$, S is called *completely continuous* if, there is a sequence of finite rank linear operators $(S_n)_{n\in\mathbb{N}}$ such that $\|S_n - S\| \to 0$ as $n \to \infty$.

$\mathscr{C}_c(\mathscr{E})$ is the set of all completely continuous linear operators on $\mathscr{E}$.

Remark 2 [13]. (i) We have $\mathscr{B}(\mathscr{E}) \subset \mathscr{C}(\mathscr{E})$.

(ii) Let $A \in \mathscr{B}(\mathscr{E})$ and $S : D(S) \subset \mathscr{E} \to \mathscr{E}$ be an unbounded linear operator. Then $S + A$ is closed if and only if S is closed.

As the classical setting, we have the following lemma.

Lemma 1 [15]. *Suppose that* $\mathbb{K} = \mathbb{Q}_p$. *Let* $S \in \Phi(\mathscr{E})$ *and* $C \in \mathscr{C}_c(\mathscr{E})$, *then* $S + C \in \Phi(\mathscr{E})$ *and* $\operatorname{ind}(S + C) = \operatorname{ind}(S)$.

Similarly to the proof of Theorem 3.1 of [4], we conclude the following theorem.

Theorem 2. *Suppose that* $\mathbb{K} = \mathbb{Q}_p$. *Let* $S \in \mathscr{C}(\mathscr{E})$. *Then*

$$\sigma_e(S) = \bigcap_{K\in\mathscr{C}_c(\mathscr{E})} \sigma(S + K).$$

As the classical setting, we have

Proposition 1 [2]. *Let $\mathscr{E}$ be an ultrametric Banach space over $\mathbb{K}$. If $S, B \in \mathscr{B}(\mathscr{E})$, then $-1 \notin \sigma(SB)$ if, and only if, $-1 \notin \sigma(BS)$.*

From Definition 2.1 of [16], we get.

Definition 10. Let $S \in \mathscr{C}(\mathscr{E})$ and $B \in \mathscr{B}(\mathscr{E})$, the resolvent set $\rho(S, B)$ of the operator pencil (S, B) of the form $S - \lambda B$ is defined by

$$\rho(S, B) = \{\lambda \in \mathbb{K} : R_\lambda(S, B) = (S - \lambda B)^{-1} \in \mathscr{B}(\mathscr{E})\}.$$

$R_\lambda(S, B)$ is called the *resolvent of the operator pencil* (S, B). The spectrum $\sigma(S, B)$ of the operator pencil (S, B) of the form $S - \lambda B$ is defined by $\sigma(S, B) = \mathbb{K}\backslash\rho(S, B)$.

From Definition 2.3 of [16], we have the following:

Definition 11. Let $S \in \mathscr{C}(\mathscr{E}), B \in \mathscr{B}(\mathscr{E})$ and $\varepsilon > 0$. The pseudospectrum $\sigma_\varepsilon(S, B)$ of the operator pencil (S, B) is defined by

$$\sigma_\varepsilon(S, B) = \sigma(S, B) \cup \{\lambda \in \mathbb{K} : \|(S - \lambda B)^{-1}\| > \varepsilon^{-1}\}.$$

The pseudoresolvent $\rho_\varepsilon(S, B)$ of (S, B) is defined by

$$\rho_\varepsilon(S, B) = \rho(S, B) \cap \{\lambda \in \mathbb{K} : \|(S - \lambda B)^{-1}\| \leq \varepsilon^{-1}\},$$

by convention $\|(S - \lambda B)^{-1}\| = \infty$ if, and only if, $\lambda \in \sigma(S, B)$.

As the proof of Theorem 2.14 of [8], the essential spectrum of closed linear operator pencils on ultrametric Banach spaces over a spherically complete field $\mathbb{K}$ is characterized by the following:

Theorem 3. *Let $\mathscr{E}$ be an ultrametric Banach space over $\mathbb{Q}_p$, let $S \in \mathscr{C}(\mathscr{E})$ and $B \in \mathscr{B}(\mathscr{E})$. Then*

$$\sigma_e(S, B) = \bigcap_{K \in \mathscr{C}_c(\mathscr{E})} \sigma(S + K, B).$$

2. Main results

We introduce the following definition.

Definition 12. Let $\mathscr{E}$ be an ultrametric Banach space over $\mathbb{K}$. Let $S \in \mathscr{C}(\mathscr{E})$, $B, C, M \in \mathscr{B}(\mathscr{E})$ and $\varepsilon > 0$. The *structured pseudospectrum* $\sigma_\varepsilon(S, M, B, C)$ of the closed linear operator pencil (S, M) is defined by

$$\sigma_\varepsilon(S, M, B, C) = \sigma(S, M) \cup \{\lambda \in \mathbb{K} : \|B(S - \lambda M)^{-1}C\| > 1/\varepsilon\}.$$

The structured pseudoresolvent $\rho_\varepsilon(S, M, B, C)$ of (S, M) is given by

$$\rho(S, M) \cap \{\lambda \in \mathbb{K} : \|B(S - \lambda M)^{-1}C\| \leq 1/\varepsilon\}.$$

By convention $\|B(S - \lambda M)^{-1}C\| = \infty$ if and only if $\lambda \in \sigma(S, M)$.

By Definition 12, we deduce the following remark.

Remark 3. If $C = B = I$, hence $\sigma_\varepsilon(S, M, I, I) = \sigma_\varepsilon(S, M)$ is the pseudospectrum of the pencil (S, M).

Theorem 4. *Let $\mathscr{E}$ be an ultrametric Banach space over $\mathbb{K}$, let $S \in \mathscr{C}(\mathscr{E})$, $B, C, M \in \mathscr{B}(\mathscr{E})$ and $\varepsilon > 0$. Then*

(i) *For all $\varepsilon_1, \varepsilon_2$ such that $\varepsilon_1 \leq \varepsilon_2$, $\sigma_{\varepsilon_1}(S, M, B, C) \subset \sigma_{\varepsilon_2}(S, M, B, C)$.*

(ii) $\sigma(S, M) = \bigcap\limits_{\varepsilon>0} \sigma_\varepsilon(S, M, B, C)$.

Proof. (i) If $\lambda \in \sigma_{\varepsilon_1}(S, M, B, C)$, hence $\|B(S-\lambda M)^{-1}C\| > \varepsilon_1^{-1} \geq \varepsilon_2^{-1}$. Then $\lambda \in \sigma_{\varepsilon_2}(S, M, B, C)$.

(ii) Since for each $\varepsilon > 0$, $\sigma(S, M) \subseteq \sigma_\varepsilon(S, M, B, C)$, then

$$\sigma(S, M) \subseteq \bigcap_{\varepsilon>0} \sigma_\varepsilon(S, M, B, C).$$

Conversely, if $\lambda \in \bigcap\limits_{\varepsilon>0} \sigma_\varepsilon(S, M, B, C)$, hence for each $\varepsilon > 0$, $\lambda \in \sigma_\varepsilon(S, M, B, C)$. If $\lambda \notin \sigma(S, M)$, hence $\lambda \in \{\lambda \in \mathbb{K} : \|B(S - \lambda M)^{-1}C\| > \varepsilon^{-1}\}$. For $\varepsilon \to 0^+$, we obtain that $\|B(S - \lambda M)^{-1}C\| = \infty$. Consequently, $\lambda \in \sigma(S, M)$. $\square$

Theorem 5. *Let $\mathscr{E}$ be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ such that $\|\mathscr{E}\| \subseteq |\mathbb{K}|$, let $S \in \mathscr{C}(\mathscr{E})$ and $B, C, M \in \mathscr{B}(\mathscr{E})$ with $0 \in \rho(B)\cap\rho(C)$ and $\varepsilon > 0$. Hence*

$$\sigma_\varepsilon(S, M, B, C) = \bigcup_{D\in\mathscr{B}(\mathscr{E}):\|D\|<\varepsilon} \sigma(S + CDB, M).$$

Proof. Firstly, we prove that

$$\bigcup_{D\in\mathscr{B}(\mathscr{E}):\|D\|<\varepsilon} \sigma(S + CDB, M) \subseteq \sigma_\varepsilon(S, M, B, C).$$

Let

$$\lambda \in \bigcup_{D\in\mathscr{B}(\mathscr{E}):\|D\|<\varepsilon} \sigma(S + CDB, M).$$

If $D = 0$, hence

$$\sigma(S, M) \subseteq \sigma_\varepsilon(S, M, B, C).$$

If $D \neq 0$. We argue by contradiction, if $\lambda \in \rho(S, M)$ and $\|B(S - \lambda M)^{-1}C\| \leq \varepsilon^{-1}$. Then for each $D \in \mathscr{B}(\mathscr{E}) : \|D\| < \varepsilon$, hence $\|DB(S - \lambda M)^{-1}C\| < 1$. Thus, $DB(S-\lambda M)^{-1}C+I$ is invertible. From Proposition 1, for each $D \in \mathscr{B}(\mathscr{E}) : \|D\| < \varepsilon$, $-1 \notin \sigma(DB(S - \lambda M)^{-1}C)$ if and only if $-1 \notin \sigma(CDB(S - \lambda M)^{-1})$. Thus

$$S + CDB - \lambda M = (I + CDB(S - \lambda M)^{-1})(S - \lambda M).$$

Hence $(S + CDB - \lambda M)^{-1} \in \mathscr{B}(\mathscr{E})$ which is a contradiction. Then

$$\bigcup_{D\in\mathscr{B}(\mathscr{E}):\|D\|<\varepsilon} \sigma(S + CDB, M) \subseteq \sigma_\varepsilon(S, M, B, C).$$

For the converse inclusion, if $\lambda \notin \sigma(S, M)$, then $\|B(S - \lambda M)^{-1}C\| > \varepsilon^{-1}$. Hence

$$\sup_{x\in\mathscr{E}\setminus\{0\}} \frac{\|B(S - \lambda M)^{-1}Cx\|}{\|x\|} > \frac{1}{\varepsilon}.$$

Consequently, there is $x \in \mathscr{E}\backslash\{0\}$ with

$$\|B(S-\lambda M)^{-1}Cx\| > \frac{\|x\|}{\varepsilon}. \tag{1}$$

Set $y = B(S-\lambda M)^{-1}Cx$, then $C^{-1}(S-\lambda M)B^{-1}y = x$. From (1), we have

$$\|C^{-1}(\lambda M - S)B^{-1}y\| < \varepsilon\|y\|. \tag{2}$$

Since $\|\mathscr{E}\| \subseteq |\mathbb{K}|$, there is $c \in \mathbb{K}\backslash\{0\}$ such that $\|y\| = |c|$, set $z = c^{-1}y$, thus $\|z\| = 1$. By (2),

$$\|(C^{-1}(\lambda M - S)B^{-1})z\| < \varepsilon. \tag{3}$$

From Theorem 1, there is $\varphi \in \mathscr{E}^*$ such that $\varphi(z) = 1$ and $\|\varphi\| = \|z\|^{-1} = 1$. Set for each $x \in \mathscr{E}$, $Dx = \varphi(x)(C^{-1}(\lambda M - S)B^{-1})z$. Then for all $x \in \mathscr{E}$,

$$\|Dx\| = |\varphi(x)|\|(C^{-1}(S-\lambda M)B^{-1})z\| \leq \|\varphi\|\|x\|\|(C^{-1}(S-\lambda M)B^{-1})z\| < \varepsilon\|x\|.$$

Hence $D \in \mathscr{B}(\mathscr{E})$ and $\|D\| < \varepsilon$. Moreover for $z \neq 0$, $Dz + (C^{-1}(S-\lambda M)B^{-1})z = 0$. Set $v = B^{-1}z \neq 0$. One can see that for $v \neq 0$, $(CDB + S - \lambda M)v = 0$. Thus

$$\lambda \in \bigcup_{D \in \mathscr{B}(\mathscr{E}):\|D\|<\varepsilon} \sigma(S + CDB, M).$$

Consequently,

$$\sigma_\varepsilon(S, M, B, C) = \bigcup_{D \in \mathscr{B}(\mathscr{E}):\|D\|<\varepsilon} \sigma(S + CDB, M). \quad \square$$

Theorem 6. *Suppose that $\mathbb{K}$ is spherically complete and $\|\mathscr{E}\| \subseteq |\mathbb{K}|$. Let $S \in \mathscr{C}(\mathscr{E})$ and $B, C, M \in \mathscr{B}(\mathscr{E})$ such that $0 \in \rho(B) \cap \rho(C)$ and $\varepsilon > 0$. Then*

$$\sigma_\varepsilon(S, M, B, C) = \sigma(S, M) \cup \{\lambda \in \mathbb{K} : \exists x \in \mathscr{E}, \|x\| = 1,\ \|C^{-1}(S-\lambda M)B^{-1}x\| < \varepsilon\}.$$

Proof. If $\lambda \in \sigma_\varepsilon(S, M, B, C)\backslash\sigma(S, M)$, then $\|B(S-\lambda M)^{-1}C\| > \varepsilon^{-1}$. Hence

$$\sup_{x \in \mathscr{E}\backslash\{0\}} \frac{\|B(S-\lambda M)^{-1}Cx\|}{\|x\|} > \frac{1}{\varepsilon}.$$

Thus there exists $x \in \mathscr{E}\backslash\{0\}$ with

$$\|B(S-\lambda M)^{-1}Cx\| > \frac{\|x\|}{\varepsilon}. \tag{4}$$

Set $y = B(S-\lambda M)^{-1}Cx \neq 0$, then $C^{-1}(S-\lambda M)B^{-1}y = x$. By (4),

$$\|C^{-1}(S-\lambda M)B^{-1}y\| < \varepsilon\|y\|. \tag{5}$$

Since $\|\mathscr{E}\| \subseteq |\mathbb{K}|$, there is $c \in \mathbb{K}\backslash\{0\}$ with $\|y\| = |c|$, put $z = c^{-1}y$, hence $\|z\| = 1$. By (5), we have

$$\|C^{-1}(S-\lambda M)B^{-1}z\| < \varepsilon. \tag{6}$$

Let $\lambda \in \mathbb{K}$ such that there exists $z \in \mathscr{E} : \|z\| = 1$ and

$$\|C^{-1}(S - \lambda M)B^{-1}z\| < \varepsilon. \tag{7}$$

From Theorem 1, there exists $\varphi \in \mathscr{E}^*$ with $\varphi(z) = 1$ and $\|\varphi\| = \|z\|^{-1} = 1$. Set for any $y \in \mathscr{E}$, $Dy = \varphi(y)(C^{-1}(\lambda M - S)B^{-1})z$. Hence for any $y \in \mathscr{E}$,

$$\|Dy\| = |\varphi(y)|\|(C^{-1}(S - \lambda M)B^{-1})z\| \leq \|\varphi\|\|y\|\|(C^{-1}(S - \lambda M)B^{-1})z\| < \varepsilon\|y\|.$$

Thus $D \in \mathscr{B}(\mathscr{E})$ and $\|D\| < \varepsilon$. Moreover for $z \neq 0$, $Dz + (C^{-1}(S - \lambda M)B^{-1})z = 0$. Set $v = B^{-1}z \neq 0$. One can see that for $v \neq 0$, $(CDB + S - \lambda M)v = 0$. Thus

$$\lambda \in \bigcup_{D \in \mathscr{B}(\mathscr{E}) : \|D\| < \varepsilon} \sigma(S + CDB, M).$$

By Theorem 5, $\lambda \in \sigma_\varepsilon(S, M, B, C)$. $\square$

The structured essential pseudospectrum of closed linear operator pencils is introduced as follows.

Definition 13. Let $\mathscr{E}$ be an ultrametric Banach space over $\mathbb{K}$, let $S \in \mathscr{C}(\mathscr{E})$, $B, C, M \in \mathscr{B}(\mathscr{E})$ and $\varepsilon > 0$. The structured essential pseudospectrum of the closed linear operator pencil (S, M) of the form $S - \lambda B$ is defined as follows:

$$\sigma_{e,\varepsilon}(S, M, B, C) = \mathbb{K} \backslash \{\lambda \in \mathbb{K} : S + CDB - \lambda M \in \Phi_0(\mathscr{E}) \text{ for all } D \in \mathscr{B}(\mathscr{E}),\ \|D\| < \varepsilon\},$$

where $\Phi_0(\mathscr{E})$ is the collection of all unbounded Fredholm operators on $\mathscr{E}$ of index 0.

We obtain the following results.

Theorem 7. *Let $\mathscr{E}$ be an ultrametric Banach space over $\mathbb{K}$. Let $S \in \mathscr{C}(\mathscr{E})$ and $B, C, M \in \mathscr{B}(\mathscr{E})$ and $\varepsilon > 0$, hence*

$$\sigma_{e,\varepsilon}(S, M, B, C) = \bigcup_{D \in \mathscr{B}(\mathscr{E}) : \|D\| < \varepsilon} \sigma_e(S + CDB, M).$$

Proof. If $\lambda \notin \sigma_{e,\varepsilon}(S, M, B, C)$, thus for each $D \in \mathscr{B}(\mathscr{E}) : \|D\| < \varepsilon$,

$$S + CDB - \lambda M \in \Phi(\mathscr{E}) \text{ and } \operatorname{ind}(S + CDB - \lambda M) = 0.$$

Hence $\lambda \notin \sigma_e(S + CDB, M)$ for each $D \in \mathscr{B}(\mathscr{E}) : \|D\| < \varepsilon$, then

$$\lambda \notin \bigcup_{D \in \mathscr{B}(\mathscr{E}) : \|D\| < \varepsilon} \sigma_e(S + CDB, M).$$

Consequently,

$$\bigcup_{D \in \mathscr{B}(\mathscr{E}) : \|D\| < \varepsilon} \sigma_e(S + CDB, M) \subseteq \sigma_{e,\varepsilon}(S, M, B, C).$$

Conversely, if

$$\lambda \notin \bigcup_{D \in \mathscr{B}(\mathscr{E}) : \|D\| < \varepsilon} \sigma_e(S + CDB, M),$$

hence for each $D \in \mathscr{B}(\mathscr{E}) : \|D\| < \varepsilon$, $\lambda \notin \sigma_e(S + CDB, M)$. Thus $S + CDB - \lambda M \in \Phi(\mathscr{E})$ and $\operatorname{ind}(S + CDB - \lambda M) = 0$ for all $D \in \mathscr{B}(\mathscr{E})$ with $\|D\| < \varepsilon$, then $\lambda \notin \sigma_{e,\varepsilon}(S, M, B, C)$. $\square$

Theorem 8. *Suppose that* $\mathbb{K} = \mathbb{Q}_p$. *Let* $S \in \mathscr{C}(\mathscr{E})$ *and* $B, C, M \in \mathscr{B}(\mathscr{E})$ *and* $\varepsilon > 0$. *Then,*

$$\sigma_{e,\varepsilon}(S, M, B, C) = \sigma_{e,\varepsilon}(S + K, M, B, C) \text{ for each } K \in \mathscr{C}_c(\mathscr{E}). \tag{8}$$

Proof. If $\lambda \notin \sigma_{e,\varepsilon}(S, M, B, C)$, hence for each $D \in \mathscr{B}(\mathscr{E})$ with $\|D\| < \varepsilon$,

$$S + CDB - \lambda M \in \Phi(\mathscr{E}) \text{ and } \operatorname{ind}(S + CDB - \lambda M) = 0.$$

From Lemma 1, for all $K \in \mathscr{C}_c(\mathscr{E})$ and $D \in \mathscr{B}(\mathscr{E})$ such that $\|D\| < \varepsilon$, we obtain

$$S + CDB + K - \lambda M \in \Phi(\mathscr{E}) \text{ and } \operatorname{ind}(S + CDB + K - \lambda M) = 0. \tag{9}$$

By (9), we get

$$\lambda \notin \sigma_{e,\varepsilon}(S + K, M, B, C).$$

Then

$$\sigma_{e,\varepsilon}(S + K, M, B, C) \subseteq \sigma_{e,\varepsilon}(S, M, B, C).$$

The opposite inclusion follows from symmetry. □

Remark 4. From Theorem 8, it follows that the structured essential pseudospectrum of closed linear operator pencils is invariant under perturbation of completely continuous linear operators on ultrametric Banach spaces over $\mathbb{Q}_p$.

Theorem 9. *Suppose that* $\mathbb{K} = \mathbb{Q}_p$ *and* $\|\mathscr{E}\| \subseteq |\mathbb{Q}_p|$. *Let* $S \in \mathscr{C}(\mathscr{E})$ *and* $B, C, M \in \mathscr{B}(\mathscr{E})$ *with* $0 \in \rho(B) \cap \rho(C)$ *and* $\varepsilon > 0$. *Then*

$$\sigma_{e,\varepsilon}(S, M, B, C) = \bigcap_{K \in \mathscr{C}_c(\mathscr{E})} \sigma_\varepsilon(S + K, M, B, C).$$

Proof. If

$$\lambda \notin \bigcap_{K \in \mathscr{C}_c(\mathscr{E})} \sigma_\varepsilon(S + K, M, B, C),$$

then there exists $K \in \mathscr{C}_c(\mathscr{E})$ with $\lambda \notin \sigma_\varepsilon(S + K, M, B, C)$. By Theorem 5, $(S + K + CDB - \lambda M)^{-1} \in \mathscr{B}(\mathscr{E})$ for any $D \in \mathscr{B}(\mathscr{E})$ such that $\|D\| < \varepsilon$. Hence

$$S + K + CDB - \lambda M \in \Phi(\mathscr{E}) \text{ and } \operatorname{ind}(S + K + CDB - \lambda M) = 0. \tag{10}$$

By Lemma 1, for each $D \in \mathscr{B}(\mathscr{E})$ with $\|D\| < \varepsilon$, we get

$$S + CDB - \lambda M \in \Phi(\mathscr{E}) \text{ and } \operatorname{ind}(S + CDB - \lambda M) = 0. \tag{11}$$

Then

$$\lambda \notin \sigma_{e,\varepsilon}(S, M, B, C).$$

Thus

$$\sigma_{e,\varepsilon}(S, M, B, C) \subseteq \bigcap_{K \in \mathscr{C}_c(\mathscr{E})} \sigma_\varepsilon(S + K, M, B, C). \tag{12}$$

Conversely, if $\lambda \notin \sigma_{e,\varepsilon}(S, M, B, C)$. From Theorem 7, for each $D \in \mathscr{B}(\mathscr{E})$ such that $\|D\| < \varepsilon$, $\lambda \notin \sigma_e(S + CDB, M)$. By Theorem 3, there is $K \in \mathscr{C}_c(\mathscr{E})$ with

$\lambda \notin \sigma(S + CDB + K, M)$, hence for each $D \in \mathscr{B}(\mathscr{E})$ such that $\|D\| < \varepsilon$, $\lambda \in \rho(S + K + CDB, M)$. Then

$$\lambda \in \bigcap_{D\in\mathscr{B}(\mathscr{E}):\|D\|<\varepsilon} \rho(S + K + CDB, M). \tag{13}$$

By Theorem 5, $\lambda \notin \sigma_\varepsilon(S + K, M, B, C)$. Consequently,

$$\lambda \notin \bigcap_{K\in\mathscr{C}_c(\mathscr{E})} \sigma_\varepsilon(S + K, M, B, C).$$

Thus

$$\sigma_{e,\varepsilon}(S, M, B, C) = \bigcap_{K\in\mathscr{C}_c(\mathscr{E})} \sigma_\varepsilon(S + K, M, B, C). \quad \square$$

Remark 5. Assume that $\mathbb{K} = \mathbb{Q}_p$ and $\|\mathscr{E}\| \subseteq |\mathbb{Q}_p|$. Let $S \in \mathscr{C}(\mathscr{E})$ and $B, C, M \in \mathscr{B}(\mathscr{E})$ with $0 \in \rho(B) \cap \rho(C)$ and $\varepsilon > 0$. From Example 3.31 of [13] and Theorem 9, we obtain

$$\sigma_{e,\varepsilon}(S, M, B, C) = \bigcap_{F\in\mathscr{F}_0(\mathscr{E})} \sigma_\varepsilon(S + F, M, B, C).$$

Proposition 2. *Let $\mathscr{E}$ be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ such that $\|\mathscr{E}\| \subseteq |\mathbb{K}|$. Let $S \in \mathscr{C}(\mathscr{E})$ and $B, C, M \in \mathscr{B}(\mathscr{E})$ such that $0 \in \rho(B) \cap \rho(C)$ and $\varepsilon > 0$. Then*

(i) $\sigma_{e,\varepsilon}(S, M, B, C) \subset \sigma_\varepsilon(S, M, B, C)$.

(ii) *For each ε_1 and ε_2 such that $0 < \varepsilon_1 < \varepsilon_2$, we get $\sigma_e(S, M) \subset \sigma_{e,\varepsilon_1}(S, M, B, C) \subset \sigma_{e,\varepsilon_2}(S, M, B, C)$.*

Proof. (i) If $\lambda \in \sigma_{e,\varepsilon}(S, M, B, C)$. From Theorem 5,

$$\lambda \in \bigcup_{D\in\mathscr{B}(\mathscr{E}):\|D\|<\varepsilon} \sigma_e(S + CDB, M).$$

By $\sigma_e(S + CDB, M) \subset \sigma(S + CDB, M)$, hence

$$\lambda \in \bigcup_{D\in\mathscr{B}(\mathscr{E}):\|D\|<\varepsilon} \sigma(S + CDB, M).$$

From Theorem 5, $\lambda \in \sigma_\varepsilon(S, M, B, C)$.

(ii) Firstly, we prove that for each $\varepsilon > 0$,

$$\sigma_e(S, M) \subset \sigma_{e,\varepsilon}(S, M, B, C).$$

If $\lambda \notin \sigma_{e,\varepsilon}(S, M, B, C)$, thus for each $D \in \mathscr{B}(\mathscr{E})$ such that $\|D\| < \varepsilon$, we get $\lambda M - (S + CDB) \in \Phi(\mathscr{E})$ and $\operatorname{ind}(\lambda M - (S + CDB)) = 0$. As $\varepsilon \to 0$, $\lambda M - S \in \Phi(\mathscr{E})$ and $\operatorname{ind}(\lambda M - S) = 0$, then $\lambda \notin \sigma_e(S, M)$. Thus

$$\sigma_e(S, M) \subset \sigma_{e,\varepsilon}(S, M, B, C).$$

Let $\lambda \notin \sigma_{e,\varepsilon_2}(S, M, B, C)$, hence for each $D \in \mathscr{B}(\mathscr{E})$ such that $\|D\| < \varepsilon_2$, $\lambda M - (S + CDB) \in \Phi(\mathscr{E})$ and $\operatorname{ind}(\lambda M - (S + CDB)) = 0$. Since $\varepsilon_1 < \varepsilon_2$, for all $D \in \mathscr{B}(\mathscr{E}) : \|D\| < \varepsilon_1$, $\lambda M - (S + CDB) \in \Phi(\mathscr{E})$ and $\operatorname{ind}(\lambda M - (S + CDB)) = 0$, thus $\lambda \notin \sigma_{e,\varepsilon 1}(S, M, B, C)$. Consequently, $\sigma_{e,\varepsilon_1}(S, M, B, C) \subset \sigma_{e,\varepsilon_2}(S, M, B, C)$. $\square$

Proposition 3. *Assume that $\mathbb{K} = \mathbb{Q}_p$ and $\|\mathscr{E}\| \subseteq |\mathbb{Q}_p|$. Let $S \in \mathscr{C}(\mathscr{E})$ and $B, C, M \in \mathscr{B}(\mathscr{E})$ with $0 \in \rho(B) \cap \rho(C)$ and $\varepsilon > 0$, hence*

$$\sigma_e(S, M) = \bigcap_{\varepsilon>0} \sigma_{e,\varepsilon}(S, M, B, C).$$

Proof. Suppose that $\lambda \in \bigcap_{\varepsilon>0} \sigma_{e,\varepsilon}(S, M, B, C)$ and $\|\mathscr{E}\| \subseteq |\mathbb{Q}_p|$. By Theorem 9,

$$\begin{aligned}\bigcap_{\varepsilon>0} \sigma_{e,\varepsilon}(S, M, B, C) &= \bigcap_{\varepsilon>0} \bigcap_{K\in\mathscr{C}_c(\mathscr{E})} \sigma_\varepsilon(S + K, M, B, C)\\ &= \bigcap_{K\in\mathscr{C}_c(\mathscr{E})} \bigcap_{\varepsilon>0} \sigma_\varepsilon(S + K, M, B, C). \qquad (14)\end{aligned}$$

From (ii) of Theorem 4,

$$\bigcap_{\varepsilon>0} \sigma_\varepsilon(S + K, M, B, C) = \sigma(S + K, M).$$

By (14), we get

$$\bigcap_{\varepsilon>0} \sigma_{e,\varepsilon}(S, M, B, C) = \bigcap_{K\in\mathscr{C}_c(\mathscr{E})} \sigma(S + K, M).$$

By Theorem 3, $\lambda \in \sigma_e(S, M)$. Conversely, $\lambda \in \sigma_e(S, M)$. From Theorem 4,

$$\lambda \in \bigcap_{K\in\mathscr{C}_c(\mathscr{E})} \sigma(S + K, M).$$

By (ii) of Theorem 4, we have

$$\lambda \in \bigcap_{\varepsilon>0} \bigcap_{K\in\mathscr{C}_c(\mathscr{E})} \sigma_\varepsilon(S + K, M, B, C).$$

Since $\|\mathscr{E}\| \subseteq |\mathbb{Q}_p|$, by Theorem 5,

$$\lambda \in \bigcap_{\varepsilon>0} \bigcup_{D\in\mathscr{B}(\mathscr{E}):\|D\|<\varepsilon} \bigcap_{K\in\mathscr{C}_c(\mathscr{E})} \sigma(S + CDB + K, M).$$

By Theorem 3, $\lambda \in \bigcap_{\varepsilon>0} \bigcup_{D\in\mathscr{B}(\mathscr{E}):\|D\|<\varepsilon} \sigma_e(S + CDB, M)$. From Theorem 7, $\lambda \in \bigcap_{\varepsilon>0} \sigma_{e,\varepsilon}(S, M, B, C)$. Consequently,

$$\sigma_e(S, M) = \bigcap_{\varepsilon>0} \sigma_{e,\varepsilon}(S, M, B, C). \quad \square$$

Example 1. Let $B, C, M \in \mathscr{B}(\mathscr{E}_\omega)$ be diagonal operators such that $0 \in \rho(M)$ and for any $i \in \mathbb{N}, Be_i = b_i e_i, Ce_i = c_i e_i$ and $Me_i = m_i e_i$ where $\sup_{i\in\mathbb{N}} |b_i|$, $\sup_{i\in\mathbb{N}} |c_i|$ and $\sup_{i\in\mathbb{N}} |m_i|$ are finite. Let S be an unbounded diagonal operator defined on $\mathscr{E}_\omega$ by for all $i \in \mathbb{N}, Se_i = a_i e_i$ where $\lim_{i\to\infty} |a_i| = \infty$ and

$$D(S) = \{x = (x_i)_{i\in\mathbb{N}} \in \mathscr{E}_\omega : \lim_{i\to\infty} |x_i||a_i|\|e_i\| = 0\}.$$

One can see that

$$\sigma(S,M) = \{a_i m_i^{-1} : i \in \mathbb{N}\},$$

and for each

$$\lambda \in \rho(S,M), \quad \|B(S-\lambda M)^{-1}C\| = \sup_{i\in\mathbb{N}} \left|\frac{b_i c_i}{a_i - \lambda m_i}\right|.$$

Consequently,

$$\sigma_\varepsilon(S,M,B,C) = \{a_i m_i^{-1} : i \in \mathbb{N}\} \cup \left\{\lambda \in \mathbb{K} : \sup_{i\in\mathbb{N}} \left|\frac{b_i c_i}{a_i - \lambda m_i}\right| > \frac{1}{\varepsilon}\right\}.$$

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