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SHARP BOUNDS ASSOCIATED WITH THE ZALCMAN CONJECTURE FOR THE INITIAL COEFFICIENTS AND SECOND HANKEL DETERMINANTS FOR CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS

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Abstract: In this paper, we obtain sharp bounds in the Zalcman conjecture for the initial coefficients, the second Hankel determinant $H_{2,2}(f) = a_2a_4 - a_3^2$ and an upper bound for the second Hankel determinant $H_{2,3}(f) = a_3a_5 - a_4^2$ for the functions belonging to a certain subclass of analytic functions. The practical tools applied in the derivation of our main results are the coefficient inequalities of the Carathéodory class $\mathscr{P}$.

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1. Introduction

Let $\mathscr{H}$ denote the class of all analytic functions defined in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathscr{A}$ represent the class of functions $f \in \mathscr{H}$ satisfying the normalized conditions namely $f(0) = f'(0) - 1 = 0$, i.e., of the form

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_1 := 1,\ z \in \mathbb{D}. \tag{1.1}$$

By S, we denote the subfamily of $\mathscr{A}$, consisting of all univalent functions (i.e., one-to-one) in $\mathbb{D}$. Pommerenke [1] characterized the n^{th} Hankel determinant of order r, for f given in (1.1) with $r, n \in \mathbb{N} = \{1, 2, 3, \dots\}$ as

$$H_{r,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+r-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+r} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+r-1} & a_{n+r} & \cdots & a_{n+2r-2} \end{vmatrix}. \tag{1.2}$$

The Fekete–Szegö functional is obtained for $r = 2$ and $n = 1$ in (1.2) and denoted by $H_{2,1}(f)$, where

$$H_{2,1}(f) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2.$$

Further, sharp bounds for the functional $|a_2a_4 - a_3^2|$ are obtained in (1.2) for $r = 2$ and $n = 2$, the Hankel determinant of order two

$$H_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2.$$

In recent years, many authors have focussed research on the estimation of an upper bound for $|H_{2,2}(f)|$. The exact estimates of $|H_{2,2}(f)|$ for the family of univalent functions, namely bounded turning, starlike and convex, denoted by $\Re$, S^* and $\mathscr{K}$, respectively, fulfilling the analytic conditions $\mathrm{Re}\{f'(z)\} > 0$, $\mathrm{Re}\{\frac{zf'(z)}{f(z)}\} > 0$ and $\mathrm{Re}\{1 + \frac{zf''(z)}{f'(z)}\} > 0$ in the unit disc $\mathbb{D}$, were proved by Janteng et al. (see [2, 3]), the bounds as 4/9, 1, and 1/8 were derived. For recent results on the second Hankel determinants (see [4–8]). Similarly, by taking $r = 2$ and $n = 3$ in (1.2), we have $H_{2,3}(f) = a_3a_5 - a_4^2$, the second Hankel determinant, for which Zaprawa [9] derived sharp bounds $|H_{2,3}(f)| \leq 1$ for the class S^* and $|H_{2,3}(f)| \leq \frac{1}{15}$ for the class $\mathscr{K}$ with the assumption that $a_2 = 0$ in f given in (1.1). By the results derived by Zaprawa [9], recently, Andy Liew Pik Hern et al. [10] have shown that $|H_{2,3}(f)| \leq \frac{13}{16}$ for $f \in S_s^*$ and $|H_{2,3}(f)| \leq \frac{13}{240}$ for $f \in \mathscr{K}_s$, where S_s^* and $\mathscr{K}_s$ denote the families of starlike and convex functions with respect to symmetric points, analytically defined as

$$f \in S_s^* \Leftrightarrow \mathrm{Re}\left\{\frac{2zf'(z)}{f(z) - f(-z)}\right\} > 0, \quad z \in \mathbb{D}. \tag{1.3}$$

$$f \in \mathscr{K}_s \Leftrightarrow \mathrm{Re}\left\{\frac{2\,\{zf'(z)\}'}{zf'(z) + zf'(-z)}\right\} > 0, \quad z \in \mathbb{D}. \tag{1.4}$$

Choosing $r = 2$ and $n = p + 1$ in (1.2), we obtain the Hankel determinant of second order for the p-valent function (see [11])

$$H_{2,p+1}(f) = \begin{vmatrix} a_{p+1} & a_{p+2} \\ a_{p+2} & a_{p+3} \end{vmatrix} = a_{p+1}a_{p+3} - a_{p+2}^2,$$

In the 1960s Zalcman posed a conjecture that if $f \in S$ then

$$\left|a_n^2 - a_{2n-1}\right| \leq (n-1)^2 \quad \text{for } n = 2, 3, \ldots; \tag{1.5}$$

the equality holds only for the Koebe function $k(z) = z/(1-z)^2$ or its rotations. For functions in S, Krushkal proved the Zalcman conjecture for $n = 3$ (see [12]) and recently for $n = 4, 5, 6, \ldots$ [13]. This remarkable conjecture was investigated by many researchers and is still an open problem for functions belonging to class S when $n > 6$. The Zalcman conjecture was proved for certain special subclasses of S, such as starlike, typically real, and close-to-convex functions (see [12, 14]). Recently, Abu Muhanna et al. [15] solved the Zalcman conjecture for the class $\mathscr{F}$ consisting of the functions $f \in \mathscr{A}$ satisfying the analytic condition

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > -1/2, \quad z \in \mathbb{D}.$$

Functions in the class $\mathscr{F}$ are known to be convex in some direction (and hence close-to-convex and univalent) in $\mathbb{D}$. In 1988, Ma [16] proved the Zalcman conjecture for close-to-convex functions. For $f \in S$, Ma [17] proposed a generalized Zalcman conjecture:

$$|a_n a_m - a_{n+m-1}| \leq (n-1)(m-1) \quad \text{for } m, n = 2, 3, \ldots, \tag{1.6}$$

which is still an open problem, and proved it for classes S^* and $S_{\mathbb{R}}$, where $S_{\mathbb{R}}$ denotes the type of all functions in $\mathscr{A}$ which are typically real. Bansal and Sokol [18] studied the Zalcman conjecture for some subclasses of analytic functions. Ravichandran and Verma [19] proved this conjecture for the classes of starlike and convex functions of a certain order and the class of functions with bounded turning. Motivated by the results mentioned above, which are associated with the Zalcman conjecture and the Hankel determinants, in the present paper, we are attempting to find sharp upper bounds for the coefficient inequalities specified in the abstract for the functions belonging to a certain subclass of analytic functions defined as follows.

Definition [20]. A mapping $f \in \mathscr{A}$ is said to be in the class $S^*\mathscr{K}_s(\beta)$ $(0 \leq \beta \leq 1)$ if

$$\operatorname{Re}\left[\frac{2\left\{zf'(z) + \beta z^2 f''(z)\right\}}{(1-\beta)\left\{f(z) - f(-z)\right\} + \beta\left\{zf'(z) + zf'(-z)\right\}}\right] > 0, \quad z \in \mathbb{D}. \tag{1.7}$$

For $\beta = 0$ and $\beta = 1$ in (1.7), we get $S^*\mathscr{K}_s(0) = S_s^*$, consisting of starlike functions with respect to symmetric points, interpreted and studied by Sakaguchi [21], and $S^*\mathscr{K}_s(1) = \mathscr{K}_s$, consisting of convex functions with respect to symmetric points, analyzed by Das and Singh [22], for which analytic conditions are given in (1.3) and (1.4).

In proving our results, the required sharp estimates specified below are given as lemmas suitable for functions possessing positive real part.

Let $\mathscr{P}$ be a class of all functions g having a positive real part in $\mathbb{D}$:

$$g(z) = 1 + \sum_{t=1}^{\infty} c_t z^t, \tag{1.8}$$

Every such a function is called Carathéodory function [23].

Lemma 1.1 [24]. *If $g \in \mathscr{P}$, then $|c_t| \leq 2$ for $t \in \mathbb{N}$; the equality is attained for the function $h(z) = \frac{1+z}{1-z}$, $z \in \mathbb{D}$.*

Lemma 1.2 [25]. *If $g \in \mathscr{P}$, then the estimate $|c_i - \mu c_j c_{i-j}| \leq 2$ holds for $i, j \in \mathbb{N} = \{1, 2, 3, \ldots\}$ with $i > j$ and $\mu \in [0, 1]$.*

From Lemma 1.2, Livingston [26] proved that $|c_i - c_j c_{i-j}| \leq 2$.

Lemma 1.3 [9]. *If $g \in \mathscr{P}$, then $\left|c_2 c_4 - c_3^2\right| \leq 4$. The inequality holds only for the functions*

$$h_1(z) = \frac{1+z^2}{1-z^2}, \quad h_2(z) = \frac{1+z^3}{1-z^3}$$

and their rotations.

Lemma 1.4 [27]. *Let $g \in \mathscr{P}$ be of the form* (1.8) *with $c_1 \geq 0$. Then*

$$2c_2 = c_1^2 + y(4 - c_1^2)$$

and

$$4c_3 = \left[c_1^3 + \{2c_1 y - c_1 y^2 + 2(1 - |x|^2)y\}(4 - c_1^2)\right],$$

for some complex valued x and y such that $|x| \leq 1$ and $|y| \leq 1$.

To obtain our results, we adopt some ideas from Libera and Zlotkiewicz [27].

2. Important Results

Theorem 2.1. *If $f \in S^*\mathscr{K}_s(\beta)$ $(0 \leq \beta \leq 1)$, then*

$$|a_2 a_3 - a_4| \leq \frac{1}{2(1+3\beta)} < (2-1)(3-1) = 2;$$

this inequality is sharp for $g_1(z) = \frac{1+z^3}{1-z^3}$.

Proof. For $f \in S^*\mathscr{K}_s(\beta)$, there exists $g \in \mathscr{P}$ such that

$$\frac{2\{zf'(z) + \beta z^2 f''(z)\}}{(1-\beta)\{f(z) - f(-z)\} + \beta\{zf'(z) + zf'(-z)\}} = g(z). \tag{2.1}$$

Putting the values for f, f' , f'' and g in (2.1), we get

$$\begin{aligned}[2(1+\beta)a_2 + 3(1+2\beta)a_3 z + 4(1+3\beta)a_4 z^2 + 5(1+4\beta)a_5 z^3 + \dots] \\ = [c_1 + \{c_2 + (1+2\beta)a_3\}z + \{c_3 + (1+2\beta)c_1 a_3\}z^2 \\ + \{c_4 + (1+2\beta)c_2 a_3 + (1+4\beta)a_5\}z^3 + \dots].\end{aligned} \tag{2.2}$$

Equating the coefficients for powers of z in (2.2), we obtain

$$a_2 = \frac{c_1}{2(1+\beta)}, \quad a_3 = \frac{c_2}{2(1+2\beta)}, \quad a_4 = \frac{(2c_3 + c_1 c_2)}{8(1+3\beta)}, \quad a_5 = \frac{(2c_4 + c_2^2)}{8(1+4\beta)}. \tag{2.3}$$

Using the values of a_2, a_3 and a_4 from (2.3), we have

$$\begin{aligned}a_2 a_3 - a_4 &= \frac{c_1 c_2}{4(1+\beta)(1+2\beta)} - \frac{(2c_3 + c_1 c_2)}{8(1+3\beta)} \\ &= -\frac{1}{4(1+3\beta)}\left(c_3 - \frac{(-2\beta^2 + 3\beta + 1)}{2(1+\beta)(1+2\beta)} c_1 c_2\right).\end{aligned}$$

Taking modulus on both sides and then applying Lemma 1.2 to the expression above, upon simplification, we obtain

$$|a_2 a_3 - a_4| \leq \frac{1}{2(1+3\beta)} < (2-1)(3-1) = 2. \quad \square$$

Remark 2.2. For the extremal function $g_1(z) = \frac{1+z^3}{1-z^3} = 1 + 2z^3 + 2z^6 + \dots$, we have $c_1 = 0$, $c_2 = 0$, and $c_3 = 2$. Hence, from (2.3) we obtain $a_2 = 0$, $a_3 = 0$, and $a_4 = \frac{c_3}{4(1+3\beta)}$.

Theorem 2.3. *If $f \in S^*\mathscr{K}_s(\beta)$ $(0 \leq \beta \leq 1)$, then*

$$\left|a_2^2 - a_3\right| \leq \frac{1}{(1+2\beta)} < (2-1)^2 = 1;$$

this inequality is sharp for $g_2(z) = \frac{1+z^2}{1-z^2}$.

Proof. Using the values of a_2 and a_3 from (2.3), we have

$$a_2^2 - a_3 = \frac{c_1^2}{4(1+\beta)^2} - \frac{c_2}{2(1+2\beta)} = -\frac{1}{2(1+2\beta)}\left(c_2 - \frac{(1+2\beta)}{2(1+\beta)^2}c_1^2\right).$$

Putting modulus on both sides in the expression above and applying Lemma 1.2, after simplifying, we get

$$|a_2^2 - a_3| \leq \frac{1}{(1+2\beta)}. \quad \square$$

Remark 2.4. For the extremal function $g_2(z) = \frac{1+z^2}{1-z^2} = 1 + 2z^2 + 2z^4 + \dots$, we have $c_1 = 0$ and $c_2 = 2$; Hence, from (2.3), we obtain $a_2 = 0$ and $a_3 = \frac{c_2}{2(1+2\beta)}$.

Theorem 2.5. *If $f \in S^*\mathscr{K}_s(\beta)$ $(0 \leq \beta \leq 1)$, then*

$$\left|a_3^2 - a_5\right| \leq \frac{1}{2(1+4\beta)} < (3-1)^2 = 4;$$

this inequality is sharp for $g_3(z) = \frac{1+z^4}{1-z^4}$.

Proof. Using the values of a_3 and a_5 from (2.3), we have

$$a_3^2 - a_5 = \frac{c_2^2}{4(1+2\beta)^2} - \frac{(2c_4 + c_2^2)}{8(1+4\beta)} = -\frac{1}{4(1+4\beta)}\left(c_4 - \frac{(-4\beta^2+4\beta+1)}{2(1+2\beta)^2}c_2^2\right). \quad (2.4)$$

Taking modulus on both sides and applying Lemma 1.2, after simplifying, we get

$$\left|a_3^2 - a_5\right| \leq \frac{1}{2(1+4\beta)}. \quad \square$$

Remark 2.6. For the extremal function

$$g_3(z) = \frac{1+z^4}{1-z^4} = 1 + 2z^4 + 2z^8 + \dots,$$

we have $c_2 = 0$ and $c_4 = 2$, therefore, from (2.3), we obtain $a_3 = 0$ and $a_5 = \frac{c_4}{4(1+4\beta)}$.

Theorem 2.7. *If $f \in S^*\mathscr{K}_s(\beta)$ $(0 \leq \beta \leq 1)$, then*

$$|H_{2,2}(f)| = |a_2a_4 - a_3^2| \leq \frac{1}{(1+2\beta)^2};$$

the inequality is sharp for the same function $g_2(z)$ as in Theorem 2.3.

Proof. Using the values of a_2, a_3, and a_4 from (2.3), for the expression $a_2a_4 - a_3^2$, we get

$$a_2a_4 - a_3^2 = \frac{1}{16(1+\beta)(1+2\beta)^2(1+3\beta)} \times \left(2(1+2\beta)^2c_1c_3 + (1+2\beta)^2c_1^2c_2 - 4(1+\beta)(1+3\beta)c_2^2\right), \quad (2.5)$$

which is equivalent to

$$a_2a_4 - a_3^2 = \frac{1}{16(1+\beta)(1+2\beta)^2(1+3\beta)}\big[d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2\big], \tag{2.6}$$

where

$$d_1 = 2(1+2\beta)^2, \quad d_2 = (1+2\beta)^2, \quad d_3 = -4(1+\beta)(1+3\beta). \tag{2.7}$$

Putting the values of c_2 and c_3 from Lemma 1.4 into the right-hand side of (2.6), we simplify it into

$$\begin{aligned} 4\big[d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2\big] &= \big[(d_1+2d_2+d_3)c_1^4 \\ &+ 2(d_1+d_2+d_3)c_1^2(4-c_1^2)x - d_1c_1^2(4-c_1^2)x^2 + d_3(4-c_1^2)^2x^2 + \\ &2d_1c_1(4-c_1^2)(1-|x|^2)y\big]. \end{aligned} \tag{2.8}$$

Taking modulus on both sides and applying the triangle inequality in the expression above, we get

$$\begin{aligned} 4\big|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2\big| &\le \big[|d_1+2d_2+d_3||c_1|^4 + 2|d_1||c_1||4-c_1^2||y| \\ + 2|d_1+d_2+d_3||c_1|^2|4-c_1^2||x| &+ \big\{(|d_1|-|d_3|)c_1^2 - 2|d_1||c_1||y| + 4|d_3|\big\}\,|4-c_1^2||x|^2\big]. \end{aligned} \tag{2.9}$$

By (2.7), we can now write

$$|d_1+2d_2+d_3| = 4\beta^2, \quad |d_1+d_2+d_3| = 1+4\beta, \tag{2.10}$$

$$\begin{aligned} &\big\{(|d_1|-|d_3|)c_1^2 - 2|d_1||c_1||y| + 4|d_3|\big\} \\ &= -(4\beta^2+8\beta+2)c_1^2 - 4(1+2\beta)^2c_1|y| + 16(1+\beta)(1+3\beta) \\ &= 2(c_1-2)\{-(2\beta^2+4\beta+1)c_1 - 4(1+\beta)(1+3\beta)\}, \\ &= 2(2-c_1)\{(2\beta^2+4\beta+1)c_1 + 4(1+\beta)(1+3\beta)\}, \quad |y|=1. \end{aligned}$$

Putting the calculated values from (2.10) and the value of d_1 from (2.7) into (2.9), after simplifying, we get

$$\begin{aligned} 2\big|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2\big| &\le \big[2\beta^2c_1^4 + 2(1+2\beta)^2c_1\big(4-c_1^2\big)|y| + (1+4\beta)c_1^2\big(4-c_1^2\big)|x| \\ &+ (2-c_1)\{(2\beta^2+4\beta+1)c_1 + 4(1+\beta)(1+3\beta)\}\big(4-c_1^2\big)|x|^2\big]. \end{aligned} \tag{2.11}$$

Applying the triangle inequality, restoring $|x|$ by ρ, with $|y| \le 1$, choosing $c_1 = c \in [0,2]$, on the right-hand side of (2.11) we obtain

$$\begin{aligned} 2\big|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2\big| &\le \big[2\beta^2c^4 + 2(1+2\beta)^2c(4-c^2) + (1+4\beta)c^2(4-c^2)\rho \\ +(2-c)\{(2\beta^2+4\beta+1)c + 4(1+\beta)(1+3\beta)\}(4-c^2)\rho^2\big] &= H(c,\rho) \quad \text{for } |x| = \rho \in [0,1]. \end{aligned} \tag{2.12}$$

Here

$$\begin{aligned} H(c,\rho) = [&2\beta^2c^4 + 2(1+2\beta)^2c(4-c^2) + (1+4\beta)c^2(4-c^2)\rho \\ &+ (2-c)\{(2\beta^2+4\beta+1)c + 4(1+\beta)(1+3\beta)\}(4-c^2)\rho^2]. \end{aligned} \tag{2.13}$$

To determine the maximum value of $H(c,\rho)$ over the rectangle $[0,1] \times [0,2]$, we consider the partial differential coefficient of $H(c,\rho)$ from (2.13) with regard to ρ given by

$$\frac{\partial H}{\partial \rho} = [(1+4\beta)c^2 + 2(2-c)\{(2\beta^2+4\beta+1)c + 4(1+\beta)(1+3\beta)\}\rho](4-c^2). \quad (2.14)$$

For $\rho \in (0,1)$, $c \in (0,2)$, and $(0 \leq \beta \leq 1)$, by (2.14), we notice that $\frac{\partial H}{\partial \rho} > 0$, which indicates that $H(c,\rho)$ turns out to be an increasing mapping of ρ, hence, its maximum value is attained on the boundary of the rectangle only, i.e., when $\rho = 1$. Therefore, for $\rho = 1$ in (2.13), after simplifying, we get

$$F(c) = H(c,1) = 4\beta^2c^4 - 8(1+2\beta)^2c^2 + 32(1+\beta)(1+3\beta), \quad (2.15)$$

$$F'(c) = 16\beta^2c^3 - 16(1+2\beta)^2c, \quad (2.16)$$

$$F''(c) = 48\beta^2c^2 - 16(1+2\beta)^2. \quad (2.17)$$

For the extreme values of $F(c)$, let $F'(c) = 0$. From (2.16), we have

$$16c\{\beta^2c^2 - (1+2\beta)^2\} = 0. \quad (2.18)$$

Now, let us discuss the following two instances.

Case 1. When $c = 0$, from (2.17), we note that

$$F''(0) = -16(1+2\beta)^2 < 0 \quad \text{for } 0 \leq \beta \leq 1.$$

Therefore, by the 2nd differentiation test at $c = 0$, $F(c)$ possesses the maximum value, which we can obtain from (2.15) as

$$\max_{0\leq c\leq 2} F(0) = 32(1+\beta)(1+3\beta). \quad (2.19)$$

Case 2. When $c \neq 0$, from (2.18), we get

$$c^2 = \frac{(1+2\beta)^2}{\beta^2}. \quad (2.20)$$

For $0 < \beta \leq 1$, from (2.20) we note that c^2 does not belong to $[0,2]$.

Now, simplifying the expressions (2.12) and (2.19), we obtain

$$\left|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2\right| \leq 16(1+\beta)(1+3\beta). \quad (2.21)$$

From (2.5) and (2.21), after simplifying, we get

$$\left|a_2a_4 - a_3^2\right| \leq \frac{1}{(1+2\beta)^2}. \quad \square \quad (2.22)$$

Remark 2.8. For the extremal function $g_2(z) = \frac{1+z^2}{1-z^2} = 1 + 2z^2 + 2z^4 + \ldots$, we have $c_1 = 0$, $c_2 = 2$, $c_3 = 0$, and $c_4 = 2$, for which from (2.3) we obtain $a_2 = 0$, $a_3 = \frac{c_2}{2(1+2\beta)}$, and $a_4 = 0$.

Remark 2.9. For $\beta = 0$ and $\beta = 1$ in (2.22), the particular results coincide with that of Rami Reddy and Vamshee Krishna [28].

Theorem 2.10. *If $f \in S^*\mathscr{K}_s(\beta)$ $(0 \le \beta \le 1)$, then*

$$|H_{2,3}(f)| = \left|a_3a_5 - a_4^2\right| \le \frac{13}{16(1+2\beta)(1+4\beta)}.$$

Proof. Using the values of a_3, a_4, and a_5 from (2.3) in $a_3a_5 - a_4^2$, we simplify it into

$$a_3a_5 - a_4^2 = \frac{1}{64}\left[\frac{(4c_2^3 + 8c_2c_4)}{(1+2\beta)(1+4\beta)} - \frac{(4c_1c_2c_3 + 4c_3^2 + c_1^2c_2^2)}{(1+3\beta)^2}\right]. \tag{2.23}$$

Rearranging the terms in (2.23), we have

$$\begin{aligned} a_3a_5 - a_4^2 = \frac{1}{64(1+2\beta)(1+4\beta)}&\left[4\left\{c_2c_4 - \frac{(1+2\beta)(1+4\beta)}{4(1+3\beta)^2}c_3^2\right\}\right. \\ &+ 4c_2\left\{c_4 - \frac{(1+2\beta)(1+4\beta)}{4(1+3\beta)^2}c_1c_3\right\} \\ + \frac{c_2^2}{(1+2\beta)(1+4\beta)}&\left.\left\{c_2 - \frac{(1+2\beta)(1+4\beta)}{(1+3\beta)^2}c_1^2\right\} + \frac{3c_2^3}{(1+2\beta)(1+4\beta)}\right]. \end{aligned} \tag{2.24}$$

Taking modulus on both sides and applying Lemmas 1.1, 1.2, and 1.3, upon simplification, we obtain

$$|H_{2,3}(f)| = \left|a_3a_5 - a_4^2\right| \le \frac{13}{16(1+2\beta)(1+4\beta)}. \quad \square \tag{2.25}$$

Remark 2.11. For $\beta = 0$ and $\beta = 1$ in (2.25), the results coincide with that of Andy Liew Pik Hern et al. [10].

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